

# ENERGY LEVELS OF STEADY STATES FOR THIN FILM TYPE EQUATIONS

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ABSTRACT. We study the phase space of the evolution equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x$$

by means of a dissipated energy (a Liapunov function). Here  $h(x, t) \geq 0$ , and at  $h = 0$  the coefficient functions  $f > 0$  and  $g$  can either degenerate to 0, or blow up to  $\infty$ , or tend to a nonzero constant.

We first show all positive periodic steady states are ‘energy unstable’ fixed points for the evolution (meaning the energy decreases under some zero-mean perturbation) if  $(g/f)'' \geq 0$  or if the perturbations are allowed to have period longer than that of the steady state.

For power law coefficients ( $f(y) = y^n$  and  $g(y) = \mathcal{B}y^m$  for some  $\mathcal{B} > 0$ ) we analytically determine the relative energy levels of distinct steady states. For example, with  $m - n \in [1, 2)$  and for suitable choices of the period and mean value, we find three fundamentally different steady states. The first is a constant steady state that is nonlinearly stable and is a local minimum of the energy. The second is a positive periodic steady state that is linearly unstable and has higher energy than the constant steady state; it is a saddle point. The third is a periodic collection of ‘droplet’ (compactly supported) steady states having lower energy than either the positive steady state or the constant one. Since the energy must decrease along every orbit, these results significantly constrain the dynamics of the evolution equation.

Our results suggest that heteroclinic connections could exist between certain of the steady states, for example from the periodic steady state to the droplet one. In a companion article we perform numerical simulations to confirm their existence.

## 1. INTRODUCTION

We study the evolution equation

$$(1) \quad h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x.$$

This is the one dimensional version of  $h_t = -\nabla \cdot (f(h)\nabla \Delta h) - \nabla \cdot (g(h)\nabla h)$ , which has been used to model the dynamics of a thin film of viscous liquid. The air/liquid interface is at height  $z = h(x, y, t) \geq 0$  and the liquid/solid interface is at  $z = 0$ . The one dimensional equation (1) applies if the liquid film is uniform in the  $y$  direction.

The fourth order term in the equation reflects surface tension effects and the second order term can reflect gravity, van der Waals interactions, thermocapillary effects or the geometry of the solid substrate, for example. Typically  $f(h) = h^3 + \beta h^p$  where  $0 < p < 3, \beta \geq 0$ , and  $g(h) \sim \pm h^m$  as  $h \rightarrow 0$ , where  $m \in \mathbb{R}$ . In certain applications  $g(h)$  changes sign at some positive  $h$ . We refer to [22, 26] for reviews of the physical and modeling literature.

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*Date:* February 1, 2008.

The extensively studied Cahn–Hilliard equation [8, 29] also has the form (1), with  $f \equiv 1$  and  $g(h) = 1 - 3h^2$ . See [1, 2, 3, 14] for further references on the Cahn–Hilliard equation.

Equations like (1) are of mathematical as well as physical interest: for example, Bertozzi and Pugh [5] conjectured that blow-up ( $\|h(\cdot, t)\|_\infty \rightarrow \infty$ ) is possible in some cases (*e.g.* if  $f(h) = h^n, g(h) = h^m$  with  $m > n + 2$ ), and they have proved [6] *finite time* blow-up for  $f(h) = h$  and  $g(h) = h^m$  when  $m \geq 3$ . In [17, §8] we related the steady states and some of their properties to this blow-up conjecture.

**Background and goals.** In [18] we proved linear stability and instability results for the positive *periodic* steady states of (1). Periodicity should not be regarded as a constraint, since if  $f, g > 0$  then positive bounded steady states must be periodic or constant, by [17, Theorem B.1]. And periodic steady states do exist for many equations of type (1), by the methods of [17, §2.2] or [13, 25], for example. Compactly supported ‘droplet’ steady states only exist, though, if  $g/f$  satisfies additional constraints [17, §2.2], and can have relatively low regularity at the contact line.

In this paper we concentrate mostly on positive periodic steady states, and on droplet steady states with zero contact angles. Our main investigative tool is the energy

$$\mathcal{E}(h(\cdot, t)) = \int_0^X \left[ \frac{1}{2} h_x(x, t)^2 - H(h(x, t)) \right] dx;$$

here  $h(x, t)$  is a smooth solution of (1) that is  $X$ -periodic in  $x$ , and  $H(y)$  satisfies  $H'' = g/f$ . This energy is strictly dissipated:  $(d/dt)\mathcal{E}(h(\cdot, t)) \leq 0$  with equality if and only if  $h$  is a steady state (see §2.1). Thus the energy is a Liapunov function for the evolution.

We address two questions about the energy landscape of the evolution (1).

1. Which steady states are local minima of the energy? Which are saddle points?
2. Among steady states having the same period and the same area (*i.e.* fluid volume), which has the lowest energy?

Answering these questions will help clarify the phase portrait of the evolution. For example, if all small zero–mean perturbations of a steady state can be shown to raise the energy, then the steady state might be asymptotically stable: it might be that every smooth solution starting from near the steady state relaxes back to the steady state as  $t \rightarrow \infty$ . But if some zero–mean perturbation *decreases* the energy, then asymptotic stability definitely fails.

Our requirement that the perturbations have zero mean seems reasonable from a physical standpoint, because it corresponds to a disturbance of the fluid that alters the profile without adding additional fluid. Mathematically it is reasonable because the evolution equation (1) preserves volume for spatially periodic solutions:  $\int h(x, t) dx = \int h(x, 0) dx$  for all time  $t$ . Thus zero–mean perturbations allow the possibility of relaxation back to the original steady state, while nonzero–mean perturbations do not.

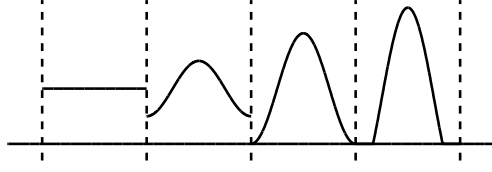


FIGURE 1. Four types of steady state.

**A sketch of definitions and results.** Take  $X > 0$ . If there is an  $X$ -periodic zero-mean perturbation  $v$  such that  $\mathcal{E}(h_{ss} + \varepsilon v) < \mathcal{E}(h_{ss})$  for all small  $\varepsilon > 0$ , then we call the steady state  $h_{ss}$  ‘energy unstable’ at period  $X$ . If instead  $\mathcal{E}(h_{ss} + \varepsilon v) > \mathcal{E}(h_{ss})$  for all sufficiently small  $\varepsilon > 0$ , for each  $X$ -periodic zero-mean perturbation  $v$ , then we call the steady state ‘energy stable’ at period  $X$ . (Some authors call this *formal* stability [16].) It is conceivable that a steady state might be energy stable and yet not be a local minimum of the energy.

Our main stability results, in Section 2, are roughly stated as follows.

- Theorem 1. For positive periodic steady states, linear instability implies energy instability. Hence our linear instability results in [18] imply that every positive  $X^*$ -periodic steady state is energy unstable at periods  $X = 2X^*, 3X^*, \dots$ , and is also energy unstable at period  $X^*$  if  $g/f$  is a strictly convex function.
- Theorems 2–3. Further, for the ‘power law’ coefficients  $f(y) = y^n$  and  $g(y) = \mathcal{B}y^m$  with  $\mathcal{B} > 0$ , we completely characterize energy stability at period  $X^*$  even when  $g/f$  is not convex, that is, when  $m - n \in (0, 1)$ .

In Section 2.6 we explain how these results for periodic boundary conditions cover the case of Neumann (‘no flux’) boundary conditions as well.

Then in Section 3 we determine the relative energy levels of three different kinds of steady state: constant steady states, positive periodic steady states, and zero contact angle droplet steady states. Figure 1 illustrates these three steady states, as well as showing a nonzero contact angle droplet steady state (about which we say little in this paper).

We have found it too difficult to determine energy levels when working with arbitrary coefficients  $f$  and  $g$ , but have obtained fairly complete answers for power law coefficients. This provides at least some insight into the general case. Further insight comes from the work of Grinfeld and Novick–Cohen [14] on the energy levels of steady states for the Cahn–Hilliard equation, which has non-power-law coefficients. The earlier work of Mischaikow [21] applies to a variety of gradient-like bistable equations.

Our main energy level results, in Section 3, for  $f(y) = y^n$  and  $g(y) = \mathcal{B}y^m$  are:

- Theorem 6. If  $m < n$  or  $m \geq n+1$  then a positive periodic steady state always has higher energy than the constant steady state with the same mean value. For  $n < m \leq n+0.75$  our analytical and numerical work suggest the positive periodic steady state has lower energy than the constant steady state.

- Theorem 9. When  $m \approx n + 0.77$  there can be two steady states with the same period and area,  $h_{ss1}$  and  $h_{ss2}$ , with  $\min_x h_{ss1}(x) < \min_x h_{ss2}(x)$ ; we essentially prove  $h_{ss2}$  is energy unstable and has higher energy than  $h_{ss1}$ , which is energy stable.
- Theorem 7. If  $m - n \in (-2, 0) \cup [1, 2)$  then a positive periodic steady state always has higher energy than a zero contact angle droplet steady state with the same mean value.
- Theorem 11. The constant steady state can have higher energy than the zero contact angle droplet steady state with the same mean value. When  $m - n \in [1, 2)$ , for example, a mountain pass scenario can occur, in which the constant steady state is a local minimum of the energy, the positive periodic steady state is an energy unstable saddle, and there is a zero contact angle droplet having lower energy than either of them.

For example, Theorem 6 with  $m = -1 < n = 3$  covers the ‘van der Waals’ case

$$(2) \quad h_t = -(h^3 h_{xxx})_x - A(h^{-1} h_x)_x$$

with  $A > 0$ . This equation has been studied by a number of other authors, *e.g.* [7, 30, 31, 33], mostly with regard to similarity solutions and film rupture (where the solution goes to zero in finite time, at some point). Our numerical work [19] on this equation pays particular attention to behavior near steady states, and how this evolves into rupture.

Our energy level results suggest possible basins of attraction around the stable steady states, and possible heteroclinic connections between steady states. In the companion article [19, §4], we investigate such possibilities with numerical simulations. For example, when  $m \approx n + 0.77$  as in Theorem 9, we find robust heteroclinic connections between the unstable positive periodic steady state  $h_{ss2}$  and the stable one  $h_{ss1}$ . For the mountain pass scenario in Theorem 11, we find that perturbing the saddle point (the periodic steady state) in one direction leads to relaxation to the constant steady state and perturbing in the opposite direction gives apparent relaxation to a droplet. A similar dichotomy was found for axisymmetric surface diffusion by Bernoff, Bertozzi and Witelski, [4, p. 744], with perturbed unduloids either relaxing to a cylinder or else pinching off in finite time.

We also present in [19, §5] simulations suggesting that small changes in the ‘mobility’ coefficient  $f$  do not break heteroclinic orbits, but can affect whether or not the solution remains positive as it evolves.

We discuss some of these conclusions and future directions further in Section 6.

**Terminology.** We write  $\mathbb{T}_X$  for a circle of circumference  $X > 0$ . As usual, one identifies functions on  $\mathbb{T}_X$  with functions on  $\mathbb{R}$  that are  $X$ -periodic and calls them *even* or *odd* according to whether they are even or odd on  $\mathbb{R}$ .

A positive periodic steady state is assumed to satisfy the steady state equation classically. A droplet steady state  $h_{ss}(x)$  (see Figures 1c and d) is by definition positive on some interval  $(a, b)$  and zero elsewhere, with  $h_{ss} \in C^1[a, b]$ ; we require  $h_{ss}$  to satisfy the steady state

equation on the open interval  $(a, b)$  only, and to have equal acute contact angles:  $0 \leq h'_{\text{ss}}(a) = -h'_{\text{ss}}(b) < \infty$ . (Throughout the paper, if a function has only one independent variable then we use  $'$  to denote differentiation with respect to that variable:  $h'_{\text{ss}} = (h_{\text{ss}})_x$ .)

We say a droplet steady state  $h_{\text{ss}}$  has ‘zero contact angle’ if  $0 = h'_{\text{ss}}(a) = -h'_{\text{ss}}(b)$ , and ‘nonzero contact angle’ otherwise. A ‘configuration’ of droplet steady states is a collection of steady droplets whose supports are disjoint. For more on the steady states and their properties, see [17].

## 2. ENERGY STABILITY FOR PERIODIC STEADY STATES

We assume throughout this section that  $f(y)$  and  $g(y)$  are  $C^2$ -smooth for  $y > 0$ , and that  $f > 0$ . Define

$$r = \frac{g}{f}.$$

Take  $X > 0$ .

We investigate stability and dynamical questions by means of a Liapunov energy. A few of the theorems follow directly from our linear stability results in [18], but most are quite different and complementary.

**2.1. Definition of the energy, and of energy instability.** The energy function for the evolution equation (1) is defined for  $\ell \in H^1(\mathbb{T}_X)$  to be

$$(3) \quad \mathcal{E}(\ell) = \int_0^X \left[ \frac{1}{2}(\ell')^2 - H(\ell) \right] dx,$$

where  $H(y)$  is a function with  $H'' = r = g/f$ .

To verify the energy  $\mathcal{E}$  is a Liapunov function for the evolution (1), suppose  $h(x, t)$  is a positive smooth solution of (1) that is  $X$ -periodic in  $x$ . Bertozzi and Pugh [5, §2] observed (generalizing [11, 12, 27]) that  $\mathcal{E}$  is dissipated by the evolution:

$$\frac{d}{dt} \mathcal{E}(h(\cdot, t)) = - \int_0^X \frac{1}{f(h(x, t))} [f(h(x, t))h_{xxx}(x, t) + g(h(x, t))h_x(x, t)]^2 dx \leq 0.$$

The dissipation is strict at each time  $t$  unless  $f(h)h_{xxx} + g(h)h_x = 0$  for all  $x$ . For smooth positive periodic solutions this occurs only when  $h(\cdot, t)$  is a steady state.

Let  $h_{\text{ss}} \in C^4(\mathbb{T}_X)$  be a positive periodic steady state of (1). It is easy to see (cf. formula (15)) that  $h_{\text{ss}}$  is a critical point for the energy  $\mathcal{E}$ , with respect to zero-mean perturbations.

**Definition.** Call  $h_{\text{ss}}$  an *energy unstable* critical point (with respect to zero-mean perturbations at period  $X$ ) if there exists a smooth  $X$ -periodic perturbation  $u(x)$  with mean value zero such that

$$\mathcal{E}(h_{\text{ss}} + \varepsilon u) < \mathcal{E}(h_{\text{ss}}) \quad \text{for all small } \varepsilon > 0.$$

That is, small perturbations in the direction  $u$  decrease the energy. (Some authors call this *formal instability* [16].)

An energy unstable steady state is necessarily a saddle point in the energy landscape, since  $\mathcal{E}$  is *increased* by the perturbation  $u(x) = \varepsilon \cos(2\pi kx/X)$  for  $k \gg 1$ .

Energy unstable steady states are not asymptotically stable in  $H^1(\mathbb{T}_X)$ , in the following sense: suppose  $h(x, t)$  is a positive smooth solution with initial data  $h_{ss} + \varepsilon u$ ; then  $h(\cdot, t) \not\rightarrow h_{ss}(\cdot)$  in  $H^1(\mathbb{T}_X)$  since for all  $t$ ,  $\mathcal{E}(h(\cdot, t)) \leq \mathcal{E}(h(\cdot, 0)) = \mathcal{E}(h_{ss} + \varepsilon u) < \mathcal{E}(h_{ss})$  (convergence in  $H^1$  would imply convergence in  $L^\infty$  and hence convergence of the energy). In fact,  $h(\cdot, t)$  cannot converge to any translate of  $h_{ss}$ , for the same reason.

The last paragraph extends to nonnegative *weak* solutions if they also dissipate the energy. Weak solutions must sometimes be considered because solutions of (1) that are initially positive might not always remain so, and where they go to zero they can lose regularity. See Bertozzi and Pugh [5, 6] for existence of nonnegative weak solutions that dissipate the energy.

**2.2. Energy instability results.** In [18, §2] we linearized the evolution equation (1) around the positive periodic steady state  $h_{ss}$  and then reduced the linear stability question to determining the sign of the first eigenvalue of a certain self-adjoint fourth order linear operator. We will not repeat the linearization here, or re-state the linear stability results of [18]. However we warn readers that when we say a steady state is ‘linearly stable’, we are including the neutrally stable case in which the first eigenvalue of the linearized operator is zero. This is unavoidable: the operator *always* has a zero eigenvalue in its spectrum, corresponding to an infinitesimal translation of the steady state in space (the evolution equation is translation invariant).

The next theorem states that if a steady state is linearly unstable then it is energy unstable. Also, we present some unstable directions, when  $g/f$  is strongly convex.

**Theorem 1.** *Let  $f, g \in C^2(0, \infty)$  with  $f > 0$ . Take  $X > 0$  and suppose  $h_{ss} \in C^4(\mathbb{R})$  is an  $X$ -periodic positive steady state of (1).*

*If  $h_{ss}$  is linearly unstable with respect to zero-mean perturbations at period  $X$ , then it is also energy unstable at period  $X$ .*

*In particular,  $h_{ss}$  is energy unstable at period  $X$  if it is non-constant and either: the least period of  $h_{ss}$  is  $X/j$  for some integer  $j \geq 2$  or else  $r = g/f$  is convex ( $r'' \geq 0$ ) and non-constant on the range of  $h_{ss}$ . For example, if  $h_{ss}$  is non-constant and  $r'' > 0$ , then  $h_{ss}$  is energy unstable in the directions  $u = \pm h'_{ss}$  and  $\pm h''_{ss}$ .*

This is proved in Section 4.1. Note for example that the theorem covers the van der Waals evolution (2), since there  $r(y) = Ay^{-1}/y^3 = Ay^{-4}$  is strongly convex. Thus all positive periodic steady states of the van der Waals evolution are energy unstable.

These energy unstable steady states (which we observed above are not asymptotically stable) are presumably *nonlinearly* unstable, in general, but we cannot prove this.

In Theorem 1 we have assumed  $f, g \in C^2(0, \infty)$ , which is generally the case for the thin film equations that are our main motivation. But our arguments are all local (involving only small perturbations of the steady state), and so the theorem still holds if the coefficient functions  $f(y)$  and  $g(y)$  are defined and  $C^2$  merely for  $y$ -values in a neighborhood of  $[h_{ssmin}, h_{ssmax}]$ .

**2.3. Review of power law steady states and their rescalings.** We now turn to *power law* coefficients:  $f(y) = y^n$  and  $g(y) = \mathcal{B}y^m$  for some exponents  $n, m \in \mathbb{R}$  and some positive constant  $\mathcal{B} > 0$ . Here

$$r(y) = \mathcal{B}y^{q-1}$$

where

$$\boxed{q := m - n + 1}$$

This exponent  $q$  determines many properties of the steady states, including (usually) their linear stability.

The evolution equation (1) becomes

$$h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x.$$

To state our results on energy stability for this power law evolution, we first review some properties of the steady states and explain how to rescale them to solutions of a canonical nonlinear oscillator ODE, given as equation (8–9) below.

We start with a non-constant positive periodic steady state  $h_{ss} \in C^4(\mathbb{T}_X)$  of the general evolution (1). The steady state condition for (1) integrates to give  $f(h_{ss})h_{ss}''' + g(h_{ss})h_{ss}' = C$  for some constant  $C$ . The least period of  $h_{ss}$  is  $X/j$  for some integer  $j \geq 1$ .

One finds that the constant  $C$  (the flux) equals zero, by dividing  $f(h_{ss})h_{ss}''' + g(h_{ss})h_{ss}' = C$  by  $f(h_{ss}) > 0$  and integrating over a period (cf. [27, 28]). Hence the steady state satisfies

$$(4) \quad h_{ss}''' + r(h_{ss})h_{ss}' = 0.$$

[If  $h_{ss}$  were a droplet steady state then again  $C = 0$ , by [17, Theorem 2.1], and equation (4) would hold wherever  $h_{ss}$  is positive.]

Integrating, the steady states have a nonlinear oscillator formulation:

$$(5) \quad h_{ss}'' + H'(h_{ss}) = 0$$

holds wherever  $h_{ss}$  is positive. Here  $H(y)$  is a function with  $H'' = r = g/f$ ; if we regard  $x$  as a ‘time’ variable then  $\frac{1}{2}h_{ss}'(x)^2 + H(h_{ss}(x))$  is a conserved quantity.

Returning to the power law evolution, remember  $r(y) = \mathcal{B}y^{q-1}$ . Thus for  $q \neq 0$  we can write the steady state equation (5) as

$$(6) \quad h''_{\text{ss}} + \frac{\mathcal{B}h_{\text{ss}}^q - D}{q} = 0$$

for some constant  $D$ . For  $q = 0$  the analogous equation is  $h''_{\text{ss}} + \mathcal{B} \log h_{\text{ss}} - D = 0$ . This oscillator equation involves three constants:  $q$ ,  $\mathcal{B}$ , and  $D$ . We remove  $\mathcal{B}$  and  $D$  by rescaling: let

$$(7) \quad k(x) = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{1/q} h_{\text{ss}} \left( \left(\frac{D}{\mathcal{B}}\right)^{1/2q} \frac{x}{D^{1/2}} \right), & q \neq 0, \\ e^{-D/\mathcal{B}} h_{\text{ss}} \left( e^{D/2\mathcal{B}} \frac{x}{\mathcal{B}^{1/2}} \right), & q = 0. \end{cases}$$

For  $q \neq 0$  this rescaling uses that  $D > 0$ , by [17, §3.1]. [A different rescaling would be used to study droplet steady states [17, §§3.2,4].] The steady state equation (6) rescales to

$$(8) \quad k'' + \frac{k^q - 1}{q} = 0, \quad q \neq 0,$$

$$(9) \quad k'' + \log k = 0, \quad q = 0.$$

Differentiating, we find for all  $q$  that  $k''' + k^{q-1}k' = 0$ , and so  $k$  satisfies  $(k^n k''' + k^m k')' = 0$ , *i.e.* it is a steady state of  $k_t = -(k^n k_{xxx})_x - (k^m k_x)_x$ .

Since  $h_{\text{ss}}$  is non-constant, positive and periodic, we know  $k''(x_0) > 0$  for some point  $x_0$ . Evaluating (8–9) at  $x_0$  shows the minimum value of  $k$  is less than 1. Also  $k'(0) = 0$  since (after a suitable translation)  $h_{\text{ss}}$  has its minimum at  $x = 0$ . Introducing the notation  $k_\alpha$  for the solution  $k$  that has minimum value  $\alpha \in (0, 1)$ , at  $x = 0$ , we have

$$(10) \quad 0 < k_\alpha(0) = \alpha < 1, \quad k'_\alpha(0) = 0.$$

Thus every steady state  $h_{\text{ss}}$  can be rescaled to a  $k_\alpha$ , as above. Conversely, for each  $q \in \mathbb{R}$  and  $\alpha \in (0, 1)$  there exists a unique smooth positive periodic  $k_\alpha$  satisfying equations (8–9) and (10) (see [17, Proposition 3.1]). The same holds for  $\alpha = 0$  when  $q > -1$ , except that  $k_0$  may be only  $C^1$ -smooth at  $x = 0$  (see [17, Theorem 3.2]). To illustrate, Figure 2 plots the steady states  $k_\alpha$  over two periods, for  $q = 3$  and eight  $\alpha$ -values between 0 and 1; see [17, §6.1] for details.

Note that the map  $(\alpha, x) \mapsto k_\alpha(x)$  is  $C^\infty$ -smooth for  $(\alpha, x) \in (0, 1) \times \mathbb{R}$ , by an ODE theorem giving smooth dependence on the initial data [15, Ch. V §4]. We write

$$P = P(\alpha) \quad \text{and} \quad A = A(\alpha)$$

for the least period of  $k_\alpha$  and for the area under its graph,  $A = \int_0^P k_\alpha(x) dx$ , respectively. Then  $P$  and  $A$  are smooth functions of  $\alpha$  that approach  $2\pi$  as  $\alpha \rightarrow 1$ , by [18, Lemma 6]. The function

$$E(\alpha) := P(\alpha)^{3-q} A(\alpha)^{q-1} = P(\alpha)^2 [A(\alpha)/P(\alpha)]^{q-1}$$



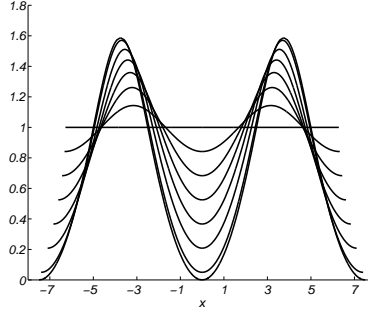


FIGURE 2. Steady states  $k_\alpha(x)$ , when  $q = 3$ .

determines whether the steady state is energy unstable or stable, in several results below.

The above rescaling ideas are a useful tool throughout the paper. We hope this tool does not obscure the fact that stability and energy level properties for equations of type (1) seem to be determined by the period map of a family of steady states  $h_{ss}$  with fixed area but varying amplitudes and periods, or alternatively the area map of a family of steady states with fixed period but varying amplitudes and areas. This is how one should think of the function  $E(\alpha)$ ; see [18, §6.3] for more on this. The same underlying idea appears in the work of Grinfeld and Novick–Cohen [14] on the Cahn–Hilliard equation, an equation which is not amenable to rescaling in the same way.

#### 2.4. Energy in/stability for the power law evolution.

**Theorem 2.** *Let  $h_{ss} \in C^4(\mathbb{R})$  be a non-constant positive  $X$ -periodic steady state of the power law evolution  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ . Translate  $h_{ss}$  to put its minimum at  $x = 0$  so that  $h_{ss}$  rescales to  $k_\alpha$  for some  $\alpha \in (0, 1)$ , as in §2.3.*

*If  $q < 1$  or  $q > 2$  then  $h_{ss}$  is energy unstable in the directions  $u = \pm h'_{ss}$  and  $u = \pm h''_{ss}$ .*

*If  $q = 2$ , or if  $q > 1$  and  $E'(\alpha) > 0$ , then  $h_{ss}$  is energy unstable.*

We prove the theorem in Section 4.2. Its first statement follows immediately from Theorem 1, since  $r(y) = \mathcal{B}y^{q-1}$  is strongly convex ( $r'' > 0$ ) when  $q < 1$  or  $q > 2$ . The final statement of the theorem certainly applies when  $q \geq 2$ , since then  $E' > 0$  by [18, Theorem 11]. Computational studies [17, §6.1] suggest  $E'(\alpha) > 0$  for all  $\alpha \in (0, 1)$  except when  $q \in [1, 1.794]$  (approximately); Figure 5 plots  $E(\alpha)$  for certain of these  $q$ -values.

To help explain the appearance of the criterion  $E'(\alpha) > 0$  for energy instability, in the theorem, we refer the reader to [18, §6.3] and to [19, §3] for a ‘bifurcation diagram’ interpretation of the function  $\alpha \mapsto E(\alpha)$  in terms of a family of steady states  $h_{ss}$  with fixed area but varying amplitudes and periods, or fixed period but varying amplitudes and areas.

The case  $1 < q < 2$  with  $E'(\alpha) \leq 0$  is not covered by Theorem 2. By [18, Theorem 9] we know  $h_{ss}$  is linearly stable in this case. We have been unable to prove *nonlinear* stability, but

we do prove in Theorem 3 below that if  $E'(\alpha) < 0$  then small perturbations of  $h_{ss}$  in every possible direction strictly increase the energy; this is consistent with nonlinear stability.

**Definition.** Let  $h_{ss} \in C^4(\mathbb{T}_X)$  be a non-constant positive periodic steady state of (1). Call  $h_{ss}$  *energy stable* (with respect to zero-mean perturbations at period  $X$ ) if for each  $u \in H^1(\mathbb{T}_X) \setminus \{0\}$  with  $\int_0^X u \, dx = 0$  we have

$$\mathcal{E}(h_{ss} + \varepsilon u) > \mathcal{E}(h_{ss}) \quad \text{for all small } \varepsilon > 0.$$

(Some authors would call this *formal* stability [16].)

A steady state might perhaps be energy stable without being a local minimum of the energy. (For example, the function  $f(x, y) = (y - x^2)(y - 3x^2)$  on  $\mathbb{R}^2$  has the property that origin is a local minimum point on each straight line through the origin, though it is not a local minimum in  $\mathbb{R}^2$ .) Another cautionary note is that the energy is insensitive to translation: in particular, a steady state and its translates have the same energy, consistent with the translation invariance of the evolution equation itself. Numerical simulations in our article [19, §4.4] demonstrate that perturbations of an energy stable steady state can evolve towards a translate of that steady state. This suggests that any asymptotic or nonlinear stability result that can be proved will hold only up to translation.

In Section 4.3 we prove:

**Theorem 3.** *Let  $h_{ss} \in C^4(\mathbb{R})$  be a non-constant positive periodic steady state of the power law equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ , with  $h_{ss}$  having least period  $X$ . Translate  $h_{ss}$  to put its minimum at  $x = 0$  so that  $h_{ss}$  rescales to  $k_\alpha$  for some  $\alpha \in (0, 1)$ , as in §2.3.*

*If  $1 < q < 2$  and  $E'(\alpha) < 0$  then  $h_{ss}$  is energy stable.*

The hypothesis  $E'(\alpha) < 0$  seems numerically to hold for all  $\alpha \in (0, 1)$  when  $1 < q \leq 1.75$ , as indicated by Figure 3 in [18].

We have no energy stability result when  $1 < q < 2$  and  $E'(\alpha) = 0$ ; fortunately it seems  $E'(\alpha) = 0$  for at most *one*  $\alpha$ -value, for each  $q$ , as shown numerically by Figures 3–5 in [18]. Theorem 9 of [18] does imply linear stability when  $E'(\alpha) = 0$ , but [18, Theorem 10(b)] shows the space of neutrally stable directions is two dimensional (rather than one dimensional as when  $E'(\alpha) < 0$ ) and this might perhaps lead to instability.

We next address the  $q = 1$  case.

**Lemma 4.** *Let  $q = 1$  (i.e.  $m = n$ ) and suppose  $h_{ss} \in C^4(\mathbb{R})$  is a non-constant positive periodic steady state of  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^n h_x)_x$  with least period  $X$ , and translate so that  $h_{ss}$  has its minimum at  $x = 0$ . Then  $h_{ss}$  is not asymptotically stable in  $H^1(\mathbb{T}_X)$  with respect to even perturbations.*

*Proof.* The steady state equation (6) with  $q = 1$  has general solution  $h_{ss}(x) = D/\mathcal{B} + c \cos(\sqrt{\mathcal{B}}x)$ , where we have used that  $h_{ss}$  has an extremum at  $x = 0$ . Hence the period is  $X = 2\pi/\sqrt{\mathcal{B}}$ , and for all small  $\varepsilon$  we see that the perturbed function  $h_{ss}(x) + \varepsilon \cos(\sqrt{\mathcal{B}}x)$  is another positive periodic steady state solution. Thus  $h_{ss}$  is not asymptotically stable in  $H^1(\mathbb{T}_X)$  with respect to even perturbations.  $\square$

The steady state is of course not asymptotically stable with respect to *general* perturbations, since one can always perturb by translating the steady state a small distance (this remark applies for all  $q$ ). However we are really not interested in such simple translational motion. Also, translational perturbations are not even permitted under the Neumann boundary conditions that we consider below, in Section 2.6.

To summarize, when  $q = 1$  the positive periodic steady states are linearly neutrally stable with respect to zero-mean perturbations of the same period, by [12] or [18, Lemma 8], and are not asymptotically stable with respect to ‘even’ perturbations, by the above lemma. Our numerical simulations in [19, §4.3] and those of [12] suggest that a wide range of small perturbations yield solutions relaxing to nearby positive periodic steady states, suggesting they are nonlinearly stable.

In the companion paper [19, §4] we illustrate the above stability and instability theorems for the power law evolution with a variety of numerical simulations. There we find not only the short time behavior suggested by the energy (in)stability results, but also some longer time limits that are suggested by the energy level results in Section 3.

**2.5. Odd perturbations.** Returning momentarily to general coefficients  $f$  and  $g$ , in Section 4.4 we prove the energy increases under *odd* perturbations, when  $r$  is concave.

**Theorem 5.** *Let  $h_{ss} \in C^4(\mathbb{R})$  be a non-constant positive periodic steady state of (1) with coefficient functions  $f, g \in C^2(0, \infty)$ ,  $f > 0$ . Suppose  $h_{ss}$  has least period  $X$ , and translate  $h_{ss}$  to put its minimum at  $x = 0$ .*

*If  $r = g/f$  is strongly concave ( $r'' < 0$ ) then for every nontrivial  $u \in H^1(\mathbb{T}_X)$  that is odd and is such that  $h_{ss} + u > 0$ , we have  $\mathcal{E}(h_{ss} + u) > \mathcal{E}(h_{ss})$ .*

The theorem is global since the perturbations are not required to be small, and it is consistent with asymptotic stability (although convergence to a *translate* of  $h_{ss}$  seems more likely than convergence to  $h_{ss}$  itself).

Theorem 5 applies in the power law case with  $1 < q < 2$ ,  $r(y) = y^{q-1}$ . Another example with strongly concave  $r$  is an equation [20, eqn. (24)] describing the dynamics of a population of aphids, for which  $f(y) = y$ ,  $g(y) = y - c$  and  $r(y) = 1 - c/y$ .

**2.6. Relation between the periodic and Neumann stability problems.** Suppose  $h_{ss}$  is an even  $X$ -periodic steady state of the evolution equation (1) with extrema at  $x =$

$0, \pm X/2, \dots$ , so that  $h'_{ss} = h'''_{ss} = 0$  at these points. As we observed at the end of [18, §2.5], linear instability of  $h_{ss}$  with respect to periodic boundary conditions on  $(-X/2, X/2)$  is equivalent to linear instability with respect to Neumann ('no flux') conditions on the half-interval  $(0, X/2)$ ; these Neumann conditions are:  $h_x = h_{xxx} = 0$  at  $x = 0, X/2$ .

The energy of a positive smooth solution is still dissipated in the case of Neumann boundary conditions, and obviously energy instability of  $h_{ss}$  in an *even* direction  $u(x)$  for the periodic problem on  $(-X/2, X/2)$  is equivalent to energy instability in the direction  $u(x)$  for the Neumann problem on  $(0, X/2)$ . (If the perturbation  $u(x)$  is even and has mean value zero on  $(-X/2, X/2)$  then it has mean value zero on  $(0, X/2)$  as well.) Similarly, 'periodic' energy stability in all even directions on  $(-X/2, X/2)$  is equivalent to 'Neumann' energy stability in all directions on  $(0, X/2)$ .

In short, for the Neumann problem on  $(0, X/2)$ , the stability result in Theorem 3 still holds, and the instability claims involving  $\pm h''_{ss}$  in Theorems 1 and 2 also still hold, since these are even functions; the claims involving  $\pm h'_{ss}$  do not carry over, since those are odd.

### 3. RELATIVE ENERGY LEVELS OF PERIODIC, CONSTANT AND DROPLET STEADY STATES

In this section we investigate the phase space of the power law equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$  by comparing the value of the energy at the positive periodic, constant and zero-angle droplet steady states. Let  $X > 0$  and recall  $q = m - n + 1$ .

**3.1. Positive periodic vs. constant steady states.** The fluid volume  $\int_0^X h(x, t) dx$  is conserved by the evolution, under periodic boundary conditions, and so the mean value

$$\bar{h} := \frac{1}{X} \int_0^X h dx$$

is constant in time. Suppose the initial data  $h(\cdot, 0)$  arises from a small zero-mean perturbation of  $h_{ss}$ . It is natural to ask whether  $h$  can converge (while staying positive and smooth) towards the constant steady state  $\bar{h}_{ss}$ , as  $t \rightarrow \infty$ . This *cannot* happen if  $\mathcal{E}(h_{ss}) < \mathcal{E}(\bar{h}_{ss})$  and also  $h(\cdot, 0)$  is close enough to  $h_{ss}$  so that  $\mathcal{E}(h(\cdot, 0)) < \mathcal{E}(\bar{h}_{ss})$ , because the energy is dissipated by the evolution.

**Theorem 6.** *Let  $h_{ss} \in C^4(\mathbb{R})$  be a non-constant positive periodic steady state of the power law equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ , with least period  $X$ . Translate  $h_{ss}$  to put its minimum at  $x = 0$  so that  $h_{ss}$  rescales to  $k_\alpha$  for some  $\alpha \in (0, 1)$ , as in §2.3.*

*If  $q \geq 2$  or  $q < 1$  then  $\mathcal{E}(h_{ss}) > \mathcal{E}(\bar{h}_{ss})$ .*

*If  $q = 1$  then  $\mathcal{E}(h_{ss}) = \mathcal{E}(\bar{h}_{ss})$ .*

*If  $1 < q < 2$  and  $E' > 0$  on  $(\alpha, 1)$  then  $\mathcal{E}(h_{ss}) > \mathcal{E}(\bar{h}_{ss})$ .*

*If  $1 < q < 2$  and  $E' < 0$  on  $(\alpha, 1)$  then  $\mathcal{E}(h_{ss}) < \mathcal{E}(\bar{h}_{ss})$ .*

The theorem is proved in Section 5.1. For  $q = -3$  it was observed numerically in [32, §3] that  $\mathcal{E}(h_{ss}) > \mathcal{E}(\overline{h_{ss}})$ .

When  $1 < q \leq 1.75$ , numerical evidence in Figure 3 of [18] suggests  $E'(\alpha) < 0$  for all  $\alpha$ . If this is true, then  $\mathcal{E}(h_{ss}) < \mathcal{E}(\overline{h_{ss}})$  by the theorem and so there can be no heteroclinic connection from  $h_{ss}$  to the constant steady state  $\overline{h_{ss}}$ ; indeed in [19, §4.4] we find numerically for  $q = 1.5$  that  $h_{ss}$  is asymptotically stable, up to translation.

For  $1.795 \leq q < 2$ , Figure 5 of [18] suggests  $E'(\alpha) > 0$  for all  $\alpha$  and so  $\mathcal{E}(h_{ss}) > \mathcal{E}(\overline{h_{ss}})$  by the theorem. Thus when  $q \geq 1.795$  or  $q < 1$ , the instability result Theorem 2 and the energy level result Theorem 6 together lead us to suspect the existence of a heteroclinic connection from  $h_{ss}$  to  $\overline{h_{ss}}$ . In Figure 3 we present numerical simulations of such heteroclinic

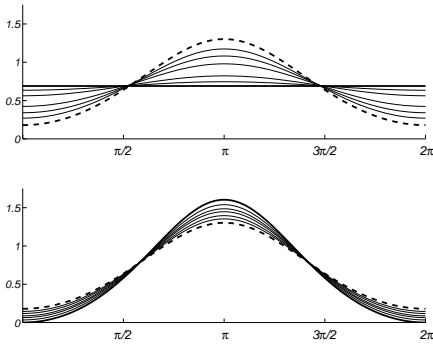


FIGURE 3.  $q = 2.5$  and  $n = 1$ , dashed line: initial data.

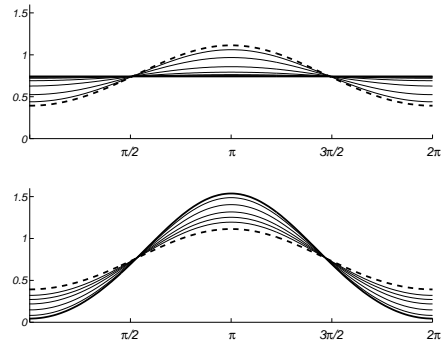


FIGURE 4.  $q = 1.768$ ,  $n = 1$ , dashed line: initial data.

orbits, taken from the companion article [19, §4.6]. The top part of the figure presents an orbit connecting the positive periodic steady state to the constant steady state. The bottom part relates to §3.2 and presents a solution connecting a perturbation of the same positive periodic steady state in finite time to a droplet profile (which will not in general be a steady state).

In particular the van der Waals equation (2) has  $q = -1 - 3 + 1 = -3$ , and so by the first claim of Theorem 6, the energy of any positive periodic steady state is greater than that of the constant steady state. This was observed numerically by Witelski and Bernoff [32, §3]. See [30, 32] and our companion paper [19, §4.1] for numerical simulations of this equation and a discussion of the literature.

For  $q \in (1.75, 1.794]$ , see the remarks around Theorem 9.

For the Cahn–Hilliard equation, the analogue of Theorem 6 (comparing energy levels of non-constant and constant steady states) can be found in Grinfeld and Novick–Cohen’s work [14, Theorem 4.1]; further, [14, §7] discusses a number of results on existence of heteroclinic connections. See also [21, §3.4].

**3.2. Positive periodic vs. droplet steady states.** We do not yet have stability results for *droplet* steady states. But here we show under certain conditions that the energy of a

zero-angle droplet steady state must be lower than that of a positive periodic steady state  $h_{ss}$  whose period exceeds the length of the droplet. In our article [19, §4] we show numerically that in these cases, the droplet seems to be strongly attracting, with heteroclinic orbits from  $h_{ss}$  towards the droplet steady state as shown in the bottom part of Figure 3.

**Theorem 7.** *Let  $h_{ss} \in C^4(\mathbb{R})$  be a non-constant positive periodic steady state of the power law equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ , with  $h_{ss}$  having least period  $X$  and area  $A_{ss} = \int_0^X h_{ss} dx$ .*

*If  $-1 < q < 1$  or  $2 \leq q < 3$ , or if  $1 < q < 2$  and  $E'(\alpha) > 0$  for all  $\alpha \in (0, 1)$ , then there exists a zero contact angle droplet steady state  $\hat{h}_{ss}$  with length  $\hat{X} < X$  and area  $\hat{A} = A_{ss}$ . Furthermore  $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(h_{ss})$ .*

We prove the theorem in Section 5.2.

#### REMARKS

1. Steady states with zero contact angle can occur only for  $q > -1$ , by [17, §2.2]. Hence we do not consider  $q \leq -1$  in the theorem; indeed variational methods show there can be droplet steady states that do not have acute contact angles [24].

2. The theorem presumably applies to  $1.795 \leq q < 2$ , since it seems numerically that  $E' > 0$  for those  $q$ -values. For  $q \geq 3$  we think a similar theorem might hold but with  $\hat{h}_{ss}$  being a *configuration* of disjoint zero-angle droplets.

3. Since  $\mathcal{E}(h_{ss}) > \mathcal{E}(\hat{h}_{ss})$ , there might be an orbit from  $h_{ss}$  to  $\hat{h}_{ss}$ . This orbit might describe a positive solution that converges to the nonnegative droplet profile as  $t \rightarrow \infty$ , or it might describe a positive solution that loses positivity in finite time and *then* approaches the droplet as a nonnegative weak solution. For  $-1 < q < 1$  and  $2 \leq q < 3$ , Theorems 6 and 7 are consistent with the idea that the unstable positive periodic steady state  $h_{ss}$  and its stable manifold form a separatrix between the basin of attraction of the constant steady state and the basin of attraction of the droplet steady state. In particular, after perturbing  $h_{ss}$  in one direction one seems to find a solution that converges to the constant steady state, while perturbing in the opposite direction often yields a solution that converges to a droplet profile. We present some numerical evidence for such behavior in Figure 3, and discuss this at length in the companion article [19, §4]. See also the ‘mountain pass’ remark after Theorem 11.

The preceding theorem and its Remarks are definitely not valid for  $q \in (1, 1.75]$ , since for these  $q$ -values we show there does not even *exist* a zero contact angle steady state with length less than  $X$ :

**Theorem 8.** *Let  $1 < q \leq 1.75$  and suppose  $E'(\alpha) < 0$  for all  $\alpha \in (0, 1)$ . Let  $h_{ss} \in C^4(\mathbb{T}_X)$  be a non-constant positive periodic steady state of the power law equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ , with  $h_{ss}$  having least period  $X$  and area  $A_{ss} = \int_0^X h_{ss} dx$ . Assume*

$\hat{h}_{ss}$  is nonnegative and piecewise- $C^1$  on  $\mathbb{T}_X$ , has area  $A_{ss}$ , and is smooth on the set where it is positive and satisfies there the ‘nonlinear oscillator’ steady state equation (6).

Then  $\hat{h}_{ss}$  is either constant or is a translate of  $h_{ss}$ , or is a configuration of nonzero contact angle droplet steady states. Specifically,  $\hat{h}_{ss}$  cannot be a zero contact angle droplet steady state.

We prove the theorem in Section 5.3. Note that the hypothesis  $E' < 0$  seems to hold for  $1 < q \leq 1.75$ , by the numerical evidence in Figure 3 of [18].

Finally, for  $q \in (1.75, 1.794]$  approximately, we know by the analytical and numerical work in [17, §5.1] that there can be *two* positive periodic steady states with the same period and area. The next theorem shows that the steady state with smaller minimum value (and larger amplitude) is energy stable, and has lower energy than the other one, which is energy unstable. This leads us to suspect there exists an orbit from the energy unstable steady state to the energy stable one, at least when the steady states have been chosen to have their minima at the same location. The bottom plot of Figure 4 presents a numerical simulation of such an orbit, taken from [19, §4.5]. The top plot of the figure presents an orbit connecting the unstable positive periodic steady state to the constant steady state, similar to the top plot of Figure 3.

**Theorem 9.** *Assume  $1 < q < 2$  and there exists  $\alpha_{crit} \in (0, 1)$  with  $E'(\alpha) < 0$  on  $(0, \alpha_{crit})$  and  $E'(\alpha) > 0$  on  $(\alpha_{crit}, 1)$ , and assume  $\alpha \mapsto \alpha P(\alpha)^{2/(q-1)}$  is strictly increasing for  $\alpha \in (0, 1)$ .*

*Suppose  $h_{ss1}$  and  $h_{ss2}$  are non-constant positive periodic steady states of the power law equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ , with  $h_{ss1}$  and  $h_{ss2}$  having the same least period  $X$  and same area  $\int_0^X h_{ss1} dx = \int_0^X h_{ss2} dx$ .*

*If  $h_{ss1}(x)$  has lower minimum value than  $h_{ss2}(x)$ , then  $h_{ss1}$  is energy stable,  $h_{ss2}$  is energy unstable, and  $\mathcal{E}(h_{ss1}) < \mathcal{E}(h_{ss2})$ . Furthermore,  $\mathcal{E}(h_{ss2}) > \mathcal{E}(\overline{h_{ss2}})$ .*

We prove this in Section 5.4. The hypothesis about  $E(\alpha)$  being first strictly decreasing and then strictly increasing is confirmed numerically for  $q$  in the interval  $(1.750, 1.794]$  by [17, §6.1]; see Figure 5. Numerical work also confirms the hypothesis about  $\alpha P(\alpha)^{2/(q-1)}$  being a strictly increasing function of  $\alpha$  for all  $q \in (1, 2)$ .

Theorem 9 is analogous to [14, Theorem 4.1(v)] for the transitional and metastable cases of the Cahn–Hilliard equation, where Grinfeld and Novick–Cohen show the energy of a monotonic ‘interface’ solution is less than that of a monotonic ‘spike’ solution having the same length and area.

**3.3. Constant steady states: stability results.** For any number  $\bar{h} > 0$  the constant function  $h_{ss} \equiv \bar{h}$  is a steady state of the general evolution equation (1). We now develop analogues for this constant steady state of our earlier stability results. Then in the following

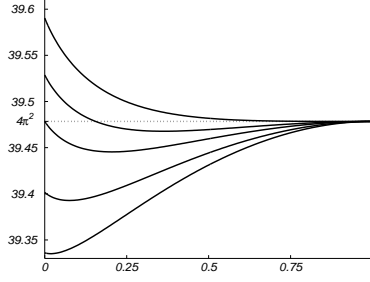


FIGURE 5. Plots of  $E(\alpha)$  for  $q = 1.75, 1.76, 1.768, 1.78, 1.79$ . Top curve:  $q = 1.75$ , the curves move down as  $q$  increases.

subsection we can compare the energies of the constant and droplet steady states, for power law coefficients.

We start by recalling a characterization of linear instability for steady states. From [18, Appendix A], an  $X$ -periodic positive steady state  $h_{ss} \in C^4(\mathbb{R})$  of (1) is linearly unstable with respect to zero-mean perturbations at period  $X$  if and only if  $\tau_1(h_{ss}) < 0$ , where

$$(11) \quad \tau_1(h_{ss}) := \min_u \frac{\int_0^X [(u')^2 - r(h_{ss})u^2] dx}{\int_0^X u^2 dx};$$

the minimum here is taken over nonzero  $u \in H^1(\mathbb{T}_X)$  with  $\int_0^X u dx = 0$ . The trial function  $u$  corresponds to a perturbation of  $h_{ss}$ .

**Theorem 10.** *Let  $f, g \in C^2(0, \infty)$  with  $f > 0$ , and write  $r = g/f$ . Let  $\bar{h}, X > 0$ . Then for the constant steady state  $h_{ss} \equiv \bar{h}$  of (1), the eigenvalue  $\tau_1$  in (11) is  $\tau_1(\bar{h}) = (2\pi/X)^2 - r(\bar{h})$ . The  $\tau_1(\bar{h})$ -eigenspace is spanned by  $\sin(2\pi x/X)$  and  $\cos(2\pi x/X)$ .*

*Thus with respect to zero-mean perturbations at period  $X$ , the constant steady state is*

$$\begin{aligned} &\text{linearly unstable} \quad \text{if } r(\bar{h})X^2 > 4\pi^2, \\ &\text{linearly neutrally stable} \quad \text{if } r(\bar{h})X^2 = 4\pi^2, \\ &\text{linearly asymptotically stable} \quad \text{if } r(\bar{h})X^2 < 4\pi^2. \end{aligned}$$

(a) *If  $r(\bar{h})X^2 > 4\pi^2$ , or if  $r(\bar{h})X^2 = 4\pi^2$  and  $r''(\bar{h}) > 0$ , then the constant steady state is energy unstable in the directions  $\pm \sin(2\pi x/X)$  and  $\pm \cos(2\pi x/X)$ .*

(b) *If  $r(\bar{h})X^2 < 4\pi^2$ , or if  $r(\bar{h})X^2 = 4\pi^2$  and  $r''(\bar{h}) < 0$ , then the constant steady state is energy stable with respect to zero-mean perturbations of period  $X$ . In fact, if  $r(\bar{h})X^2 < 4\pi^2$  then the constant steady state is a strict local minimum of the energy with respect to zero-mean perturbations in  $H^1(\mathbb{T}_X)$ , and  $\bar{h}$  is nonlinearly stable under the evolution (1), in an  $H^1(\mathbb{T}_X)$ -sense made precise in the proof.*

We prove the theorem in Section 5.5. Its linear stability assertions are well-known and are included for the sake of completeness.



Goldstein, Pesci and Shelley [12, §IIIB] used the energy to prove nonlinear instability of the constant steady state for the  $q = 1$  case ( $f(y) = y^n, g(y) = \mathcal{B}y^n, X = 2\pi$ ) with either  $2 \leq \mathcal{B} < 4$  or  $\mathcal{B} = j^2$  for some integer  $j \geq 2$ .

**3.4. Constant vs. droplet steady states.** Consider power law coefficients, so that  $r(y) = \mathcal{B}y^{q-1}$ ; then the previous theorem shows that the stability of the constant steady state  $\bar{h}$  is determined by whether the quantity  $\mathcal{B}\bar{h}^{q-1}X^2$  is  $>, =, < 4\pi^2$ .

Fix  $X > 0$ . Does a zero-angle droplet steady state exist with length at most  $X$  and with the same area  $\bar{h}X$  as the constant steady state? If such a droplet steady state exists, can it have lower energy than the constant steady state?

In this direction, in Section 5.6 we prove:

**Theorem 11.** *Let  $\bar{h}, X > 0$ , and consider the constant steady state  $h_{ss} \equiv \bar{h}$  of the power law evolution equation  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ .*

*(a) Suppose  $-1 < q < 3$ . Then there exists a zero contact angle droplet steady state  $\hat{h}_{ss}$  of length  $\hat{X} \leq X$  and area  $\bar{h}X$  if and only if*

$$(12) \quad \mathcal{B}\bar{h}^{q-1}X^2 \geq E(0) =: E_0(q).$$

*If such a droplet steady state exists, then  $\mathcal{E}(\bar{h}) > \mathcal{E}(\hat{h}_{ss})$  if and only if*

$$(13) \quad \mathcal{B}\bar{h}^{q-1}X^2 > A(0)^2 \left[ \frac{3+q}{(3-q)(q+1)} \right]^{(3-q)/q} =: L(q) \quad (\text{for } -1 < q < 3, q \neq 0)$$

*or  $\mathcal{B}\bar{h}^{-1}X^2 > 4e^2\pi/3 =: L(0)$  (for  $q = 0$ ).*

*(b) For  $q = 3$ , such a steady state  $\hat{h}_{ss}$  exists if and only if  $\mathcal{B}\bar{h}^2X^2 = E(0)$ . For  $q > 3$ ,  $\hat{h}_{ss}$  exists if and only if  $\mathcal{B}\bar{h}^{q-1}X^2 \leq E(0)$ . For all  $q \geq 3$ , if  $\hat{h}_{ss}$  exists then  $\mathcal{E}(\bar{h}) < \mathcal{E}(\hat{h}_{ss})$ .*

To understand conditions (12) and (13) see the plots of  $E_0(q)$  and  $L(q)$  in Figures 6 and 7 (constructed using the formulas for  $A(0), P(0), E(0)$  in [17, §3.1.2]). The graphs of  $E_0$  and  $L$

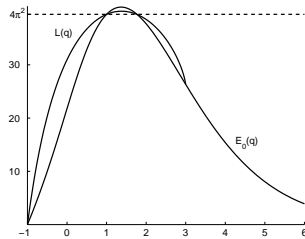


FIGURE 6. Plots of  $E_0(q)$  and  $L(q)$ .

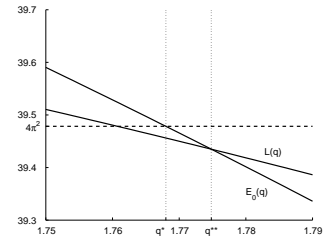


FIGURE 7. Close-up view.

intersect at  $q = -1, 1, 3$  and at  $q^{**} \approx 1.775$ . For  $1 < q < q^{**}$  the figure suggests  $E_0 > L$ , and so if the droplet steady state in Theorem 11(a) exists then it certainly has lower energy than the constant steady state. On the other hand, it appears that  $L > E_0$  when  $-1 < q < 1$  and when  $q^{**} < q < 3$ ; in these cases the energy condition (13) may or may not hold when the

existence condition (12) holds, so that the energy of the droplet steady state (if it exists) might be higher or lower than that of the constant steady state.

The dashed line at height  $4\pi^2$  in Figures 6 and 7 intersects  $E_0(q)$  at 1 and  $q^* \approx 1.768$ , and intersects  $L(q)$  at 1 and 1.761 (approx.). This line matters because the constant steady state is a strict local minimum for the energy when  $\mathcal{B}\bar{h}^{q-1}X^2 < 4\pi^2$ , by Theorem 10(b). For example, suppose that  $2 \leq q < 3$  and  $\bar{h}$  and  $X$  are such that  $E_0(q) < L(q) < \mathcal{B}\bar{h}^{q-1}X^2 < 4\pi^2$ . Then the constant steady state  $\bar{h}$  is a strict local minimum of the energy but is *not* a global minimum since  $\mathcal{E}(\bar{h}) > \mathcal{E}(\hat{h}_{ss})$  by Theorem 11(a). A mountain pass idea then suggests the energy might have a saddle point at which its value is greater than  $\mathcal{E}(\bar{h})$ . Such a saddle ought to be an energy unstable positive periodic steady state, and should have period  $X$  and area  $\bar{h}X$ . In fact we already know such a positive periodic steady state exists, by [18, Theorem 12] and Theorem 6 of this paper; we illustrate it in Figure 3. Perturbing from the saddle in one direction leads to relaxation to the constant (see the top part of Figure 3), while perturbing in the opposite direction yields apparent relaxation to a droplet (the bottom part of Figure 3). The companion article [19] contains more extensive numerical investigations.

#### 4. PROOFS OF THEOREMS 1–5

**4.1. Proof of Theorem 1.** Recall that the positive periodic steady state  $h_{ss}$  satisfies equations (4) and (5):

$$(14) \quad h_{ss}''' + r(h_{ss})h_{ss}' = 0, \quad h_{ss}'' + H'(h_{ss}) = \text{const.},$$

where  $H''(y) = r(y)$  is defined for  $y > 0$ .

We compute the first four variations of the energy (3). Let  $u \in H^1(\mathbb{T}_X)$  have mean value zero. For all  $\varepsilon$  small enough that  $h_{ss} + \varepsilon u > 0$  (so that  $H(h_{ss} + \varepsilon u)$  makes sense), we have

$$(15) \quad \left. \frac{d}{d\varepsilon} \mathcal{E}(h_{ss} + \varepsilon u) \right|_{\varepsilon=0} = - \int_0^X [h_{ss}'' + H'(h_{ss})] u \, dx = 0$$

by (14), since  $u$  has mean value zero,

$$(16) \quad \begin{aligned} \left. \frac{d^2}{d\varepsilon^2} \mathcal{E}(h_{ss} + \varepsilon u) \right|_{\varepsilon=0} &= \int_0^X [(u')^2 - r(h_{ss})u^2] \, dx \\ &= \text{numerator of Rayleigh quotient (11) for } \tau_1(h_{ss}), \end{aligned}$$

$$(17) \quad \left. \frac{d^3}{d\varepsilon^3} \mathcal{E}(h_{ss} + \varepsilon u) \right|_{\varepsilon=0} = - \int_0^X r'(h_{ss})u^3 \, dx,$$

$$(18) \quad \left. \frac{d^4}{d\varepsilon^4} \mathcal{E}(h_{ss} + \varepsilon u) \right|_{\varepsilon=0} = - \int_0^X r''(h_{ss})u^4 \, dx.$$

If the steady state  $h_{ss}$  is linearly unstable with respect to zero-mean perturbations at period  $X$ , then the numerator of the Rayleigh quotient (11) is negative for some zero-mean trial function  $u \in H^1(\mathbb{T}_X) \setminus \{0\}$ . We can assume  $u \in C^\infty(\mathbb{T}_X)$ . From (16), the second

variation of the energy in the direction  $u$  is negative, so that  $h_{ss}$  is energy unstable in the direction  $u$  as desired.

From Theorems 1 and 3 in [18],  $h_{ss}$  is linearly unstable with respect to zero-mean perturbations at period  $X$  if it is non-constant and: either the least period of  $h_{ss}$  is  $X/j$  for some integer  $j \geq 2$ , or  $r = g/f$  is convex ( $r'' \geq 0$ ) and non-constant on the range of  $h_{ss}$ . Hence  $h_{ss}$  is energy unstable in those situations.

Now assume  $h_{ss}$  is non-constant and  $r$  is strongly convex ( $r'' > 0$ ). We consider the second variation of  $\mathcal{E}$  in the direction  $u = \pm h_{ss}''$ :

$$\begin{aligned}
 \left. \frac{d^2}{d\varepsilon^2} \mathcal{E}(h_{ss} \pm \varepsilon h_{ss}'') \right|_{\varepsilon=0} &= \int_0^X [h_{ss}'''^2 - r(h_{ss}) h_{ss}''^2] dx \\
 &= - \int_0^X r(h_{ss}) [h_{ss}' h_{ss}''' + h_{ss}''^2] dx \quad \text{by (14)} \\
 &= - \int_0^X r(h_{ss}) [h_{ss}' h_{ss}'']' dx = \int_0^X r'(h_{ss}) h_{ss}'^2 h_{ss}'' dx \\
 (19) \quad &= \frac{1}{3} \int_0^X r'(h_{ss}) [h_{ss}'^3]' dx = -\frac{1}{3} \int_0^X r''(h_{ss}) h_{ss}'^4 dx.
 \end{aligned}$$

By assumption,  $r''(h_{ss}(x)) > 0$  for all  $x$ . Since  $h_{ss}'$  is not identically zero, the second variation in the direction  $u = \pm h_{ss}''$  is negative, and so  $\pm h_{ss}''$  is an energy unstable direction for  $h_{ss}$ .

It remains to prove  $h_{ss}$  is energy unstable in the directions  $u = \pm h_{ss}'$ . Here the second variation is zero, since (16) becomes

$$\int_0^X [(h_{ss}'')^2 - r(h_{ss})(h_{ss}')^2] dx = - \int_0^X [h_{ss}''' + r(h_{ss})h_{ss}'] h_{ss}' dx = 0 \quad \text{by parts and (14)}.$$

The third variation is zero for  $u = \pm h_{ss}'$  by (17), because  $h_{ss}$  is even about its minimum point while  $(\pm h_{ss}')^3$  is odd, by uniqueness for the nonlinear oscillator equation  $h_{ss}'' + H'(h_{ss}) = \text{const}$  in (14). The fourth variation is negative by (18), because  $r''(h_{ss}(x)) > 0$  by assumption. Hence  $h_{ss}$  is energy unstable in the directions  $u = \pm h_{ss}'$ , completing the proof.  $\square$

A number of other authors, working on closely related topics, have noted that translation invariance of the evolution implies the second variation of the energy in the direction  $h_{ss}'$  is zero [9, 10, 23]. Those authors then proved energy instability in the direction  $u = h_{ss}' + \eta$  for some small function  $\eta$ . Their arguments relied either on  $h_{ss}$  having least period  $X/j$  for some  $j \geq 2$  or else they did not impose a zero-mean requirement on the perturbation. Above, we have used instead the assumption  $r'' > 0$ .

#### 4.2. Proof of Theorem 2.

When  $q < 1$  or  $q > 2$ , Theorem 2 follows from the last statement of Theorem 1 since  $r(y) = \mathcal{B}y^{q-1}$  is strongly convex.

When  $q = 2$  and  $u = -h''_{ss}$ , formula (19) shows the second variation of the energy is zero, since  $r(y) = \mathcal{B}y$  and  $r'' \equiv 0$ . The third variation is negative because

$$\begin{aligned}
\left. \frac{d^3}{d\varepsilon^3} \mathcal{E}(h_{ss} - \varepsilon h''_{ss}) \right|_{\varepsilon=0} &= \mathcal{B} \int_0^X (h''_{ss})^3 dx \quad \text{by (17)} \\
&= -2\mathcal{B} \int_0^X h'_{ss} h''_{ss} h'''_{ss} dx \quad \text{by parts} \\
&= 2\mathcal{B}^2 \int_0^X h_{ss} (h'_{ss})^2 h''_{ss} dx \quad \text{since } h'''_{ss} = -\mathcal{B} h_{ss} h'_{ss} \text{ by (14)} \\
&= -\frac{2}{3} \mathcal{B}^2 \int_0^X (h'_{ss})^4 dx < 0.
\end{aligned}$$

Thus the steady state is energy unstable in the direction  $u = -h''_{ss}$ .

Now suppose  $q > 1$  and  $E'(\alpha) > 0$ . To obtain the unstable direction  $u$ , start by rescaling  $k_\alpha$  to give

$$(20) \quad K_\alpha(x) := \frac{P(\alpha)}{A(\alpha)} k_\alpha(P(\alpha)x).$$

By construction,  $K_\alpha$  has period 1 and mean value 1. Then define

$$\kappa_\alpha(x) := \frac{\partial}{\partial \alpha} K_\alpha(x);$$

$\kappa_\alpha$  is well-defined and smooth because  $P$  and  $A$  depend smoothly on  $\alpha$  while  $k_\alpha(x)$  is jointly smooth in  $(\alpha, x)$ . Notice  $\kappa_\alpha$  is even in  $x$ , has period 1, and has mean value zero:

$$\int_0^1 \kappa_\alpha(x) dx = \frac{\partial}{\partial \alpha} \int_0^1 K_\alpha(x) dx = \frac{\partial}{\partial \alpha}(1) = 0.$$

(See [18, §5.4] for more properties of  $\kappa_\alpha$ .) Let  $u(x) = \pm \kappa_\alpha(x/X)$ .

We want to show  $u$  is an unstable direction. First we show that composing the rescalings of  $h_{ss}$  to  $k_\alpha$  and then  $k_\alpha$  to  $K_\alpha$  yields

$$(21) \quad \mathcal{B}X^2 h_{ss}(xX)^{q-1} = EK_\alpha(x)^{q-1}.$$

Indeed, from the definition (20) of  $K_\alpha$ , the righthand side of (21) reduces to  $P^2 k_\alpha(Px)^{q-1}$ , and then one can substitute for  $k_\alpha$  in terms of  $h_{ss}$  using (7). Next one obtains the lefthand side of (21) by using the relation  $P = (\mathcal{B}/D)^{1/2q} D^{1/2} X$  which relates the periods of  $k_\alpha$  and  $h_{ss}$  ( $P$  and  $X$ , respectively).

Now that we have (21), use (16) to compute the second variation of the energy in the direction  $u(x) = \pm \kappa_\alpha(x/X)$  as

$$\begin{aligned}
\int_0^X [\kappa'_\alpha(x/X)^2/X^2 - \mathcal{B}h_{ss}(x)^{q-1} \kappa_\alpha(x/X)^2] dx &= \int_0^1 [\kappa'_\alpha(x)^2 - \mathcal{B}X^2 h_{ss}(xX)^{q-1} \kappa_\alpha(x)^2] dx/X \\
&= \int_0^1 [(\kappa'_\alpha)^2 - EK_\alpha^{q-1} \kappa_\alpha^2] dx/X \quad \text{by (21)} \\
&< 0
\end{aligned}$$

by the proof of [18, Prop. 14], which uses both  $q > 1$  and  $E'(\alpha) > 0$ . Thus the steady state is energy unstable in the direction  $u = \pm \kappa_\alpha(x/X)$ .  $\square$

**4.3. Proof of Theorem 3.** Since  $E'(\alpha) < 0$  by hypothesis, Proposition 15 of [18] implies  $\mu_1(\alpha) \geq 0$ , where

$$\mu_1(\alpha) := \min \left\{ \frac{\int_0^1 [(v')^2 - E(\alpha)K_\alpha^{q-1}v^2] dx}{\int_0^1 v^2 dx} : v \in H^1(\mathbb{T}_1) \setminus \{0\}, \int_0^1 v(x) dx = 0 \right\}.$$

(In fact  $E'(\alpha) \leq 0$  would suffice to get this.) Notice  $\mu_1(\alpha) = X^2\tau_1(h_{ss})$ , by letting  $v(x) = u(xX)$  and using the identity (21) and the definition (11) of  $\tau_1$ . Hence  $\tau_1(h_{ss}) \geq 0$ .

Consider  $u \in H^1(\mathbb{T}_X) \setminus \{0\}$  with mean value zero. The first variation of  $\mathcal{E}$  in the direction  $u$  is zero by (15), and the second variation of  $\mathcal{E}$  in (16) is nonnegative because it equals the numerator of the Rayleigh quotient for  $\tau_1(h_{ss})$ .

If the second variation is positive then  $h_{ss}$  is energy stable in the direction  $h_{ss}$  and we are done. If the second variation is zero then the Rayleigh quotient of  $u$  is zero, and so  $\tau_1(h_{ss}) = 0$  and  $u$  minimizes the Rayleigh quotient for  $\tau_1$  in (11). Hence  $u$  satisfies the Euler–Lagrange condition  $u'' + r(h_{ss})u = \text{const}$ , and so it satisfies  $\mathcal{L}u = 0$  where  $\mathcal{L}$  is the linearized operator defined in [18, eq. (4)] (take  $a = 0$  there). Theorem 10(a) in [18] and the hypothesis  $E'(\alpha) < 0$  now imply  $u$  is a multiple of  $h'_{ss}$ , and so  $u$  is odd. Therefore the third variation (17) of  $\mathcal{E}$  in the direction  $u$  is zero, because  $u$  is odd and  $h_{ss}$  is even. The fourth variation is positive by (18) because  $r(y) = \mathcal{B}y^{q-1}$  and  $r''(y) < 0$  for  $1 < q < 2$ . Thus  $h_{ss}$  is energy stable in the direction  $u$ .  $\square$

**4.4. Proof of Theorem 5.** As usual, the first variation of  $\mathcal{E}$  at  $h_{ss}$  is zero by (15). We now prove non-negativity of the second variation, given by (16).

First note that  $h_{ss}$  is symmetric about every point at which  $h'_{ss} = 0$ , by uniqueness for the ODE  $h''_{ss} + H'(h_{ss}) = \text{const}$  (see (14)); the uniqueness theorem applies here since the coefficient function  $H'$  is  $C^1$  (even  $C^3$ ) on the range of the positive bounded function  $h_{ss}$ .

Since  $h_{ss}$  has a minimum at  $x = 0$  by hypothesis, we conclude  $h_{ss}$  is even and  $h'_{ss} > 0$  on  $(0, X/2)$  (otherwise  $h_{ss}$  would have period less than  $X$ ).

Consider a minimizer of the Rayleigh quotient (11) with respect to *odd* functions  $u$ ; it is a smooth odd function  $\tilde{u}$  satisfying  $\tilde{u}'' + r(h_{ss})\tilde{u} + \tilde{\tau}\tilde{u} = 0$  for some constant  $\tilde{\tau}$ . Since  $\tilde{u}(0) = 0$ , one must have  $\tilde{u}'(0) \neq 0$  because otherwise  $\tilde{u} \equiv 0$  by the uniqueness theorem for linear ODEs. Also,  $\tilde{u}(X/2) = 0$  by the oddness and periodicity of  $\tilde{u}$ , and so there is a point  $b \in (0, X/2]$  with  $\tilde{u}(b) = 0$  and  $\tilde{u} \neq 0$  between 0 and  $b$ . Assume  $\tilde{u} > 0$  between 0 and  $b$

(otherwise consider  $-\tilde{u}$ ). Then

$$\begin{aligned}
\tilde{\tau} \int_0^b \tilde{u} h'_{\text{ss}} dx &= - \int_0^b [\tilde{u}'' + r(h_{\text{ss}}) \tilde{u}] h'_{\text{ss}} dx && \text{since } \tilde{u}'' + r(h_{\text{ss}}) \tilde{u} + \tilde{\tau} \tilde{u} = 0 \\
&= -\tilde{u}'(b) h'_{\text{ss}}(b) - \int_0^b [h'''_{\text{ss}} + r(h_{\text{ss}}) h'_{\text{ss}}] \tilde{u} dx && \text{by parts, since } h'_{\text{ss}}(0) = 0 \\
&= -\tilde{u}'(b) h'_{\text{ss}}(b) && \text{by (14)} \\
&\geq 0,
\end{aligned}$$

because  $\tilde{u}'(b) \leq 0$  and  $h'_{\text{ss}}(b) \geq 0$ . Since  $\tilde{u}$  and  $h'_{\text{ss}}$  are positive on  $(0, b)$  it follows that  $\tilde{\tau} \geq 0$ . Hence the second variation of  $\mathcal{E}$  in (16) is nonnegative, as desired.

The third variation at  $\varepsilon = 0$  is zero by (17), since  $u$  is odd and  $h_{\text{ss}}$  is even.

The fourth variation (18) is positive not just at  $\varepsilon = 0$  but at all  $\varepsilon \in (0, 1)$ :

$$\frac{d^4}{d\varepsilon^4} \mathcal{E}(h_{\text{ss}} + \varepsilon u) = - \int_0^X r''(h_{\text{ss}} + \varepsilon u) u^4 dx > 0$$

by the strong concavity of  $r$ , provided  $u \not\equiv 0$ . Taylor's theorem completes the proof, since for some  $\tilde{\varepsilon} \in (0, 1)$ ,

$$\mathcal{E}(h_{\text{ss}} + u) = \mathcal{E}(h_{\text{ss}}) + \frac{1}{2!} \frac{d^2}{d\varepsilon^2} \mathcal{E}(h_{\text{ss}} + \varepsilon u) \Big|_{\varepsilon=0} + \frac{1}{4!} \frac{d^4}{d\varepsilon^4} \mathcal{E}(h_{\text{ss}} + \varepsilon u) \Big|_{\varepsilon=\tilde{\varepsilon}} > \mathcal{E}(h_{\text{ss}}).$$

□

## 5. PROOFS OF THEOREMS 6—11

**5.1. Proof of Theorem 6.** We start by relating  $\mathcal{E}(h_{\text{ss}})$  to  $\mathcal{E}(k_\alpha)$ . Take

$$(22) \quad H(y) := \begin{cases} \frac{1}{q} \left[ \frac{y^{q+1}}{q+1} - y \right], & q \neq 0, -1, \\ y \log y - y, & q = 0, \\ y - \log y, & q = -1, \end{cases}$$

so that  $H''(y) = y^{q-1}$ , and recall from the definition (3) of the energy that

$$(23) \quad \mathcal{E}(h_{\text{ss}}) = \int_0^X \left[ \frac{1}{2} (h'_{\text{ss}})^2 - \mathcal{B} H(h_{\text{ss}}) \right] dx \quad \text{and} \quad \mathcal{E}(k_\alpha) = \int_0^{P(\alpha)} \left[ \frac{1}{2} (k'_\alpha)^2 - H(k_\alpha) \right] dx.$$

Denote the period of  $h_{\text{ss}}$  by  $P_{\text{ss}} = X$ , and the area by  $A_{\text{ss}} = \int_0^X h_{\text{ss}} dx$ . Writing  $P = P(\alpha)$  and  $A = A(\alpha)$ , the rescaling (7) implies

$$(24) \quad P = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{1/2q} D^{1/2} P_{\text{ss}}, & q \neq 0, \\ e^{-D/2\mathcal{B}} \mathcal{B}^{1/2} P_{\text{ss}}, & q = 0, \end{cases} \quad \text{and} \quad A = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{3/2q} D^{1/2} A_{\text{ss}}, & q \neq 0, \\ e^{-3D/2\mathcal{B}} \mathcal{B}^{1/2} A_{\text{ss}}, & q = 0. \end{cases}$$

Notice that the rescaling (7) can be written as

$$(25) \quad h_{\text{ss}}(x) = \frac{A_{\text{ss}}}{A} \frac{P}{P_{\text{ss}}} k_\alpha \left( \frac{P}{P_{\text{ss}}} x \right).$$

From (24) we obtain the invariance relation

$$(26) \quad \mathcal{B}P_{\text{ss}}^{3-q}A_{\text{ss}}^{q-1} = P^{3-q}A^{q-1} = E(\alpha),$$

and this implies

$$(27) \quad \left(\frac{A_{\text{ss}}}{A}\right)^2 \left(\frac{P}{P_{\text{ss}}}\right)^3 = \mathcal{B} \left(\frac{A_{\text{ss}}}{A}\right)^{q+1} \left(\frac{P}{P_{\text{ss}}}\right)^q.$$

Using (25), (26), (27) and the definitions (23), we at last deduce a relation between  $\mathcal{E}(h_{\text{ss}})$  and  $\mathcal{E}(k_{\alpha})$ :

$$(28) \quad \mathcal{B}^{-1} \overline{h_{\text{ss}}}^{-(q+1)} \frac{\mathcal{E}(h_{\text{ss}}) - \mathcal{E}(\overline{h_{\text{ss}}})}{P_{\text{ss}}} = \overline{k_{\alpha}}^{-(q+1)} \frac{\mathcal{E}(k_{\alpha}) - \mathcal{E}(\overline{k_{\alpha}})}{P(\alpha)},$$

where the mean values are  $\overline{h_{\text{ss}}} := A_{\text{ss}}/P_{\text{ss}}$  and  $\overline{k_{\alpha}} := A(\alpha)/P(\alpha)$ . (When checking (28), one can omit the linear terms in  $H(y)$  from the calculations, since  $h_{\text{ss}}$  and  $\overline{h_{\text{ss}}}$  have the same mean value, as do  $k_{\alpha}$  and  $\overline{k_{\alpha}}$ .)

In view of (28), then, Theorem 6 follows from:

**Proposition 12.** *Fix  $\alpha_1 \in (0, 1)$ .*

*If  $q \geq 2$  or  $q < 1$  then  $\mathcal{E}(k_{\alpha}) > \mathcal{E}(\overline{k_{\alpha}})$  for all  $\alpha \in (0, 1)$ .*

*If  $q = 1$  then  $\mathcal{E}(k_{\alpha}) = \mathcal{E}(\overline{k_{\alpha}})$  for all  $\alpha \in (0, 1)$ .*

*If  $1 < q < 2$  and  $E'(\alpha) > 0 \forall \alpha \in (\alpha_1, 1)$  then  $\mathcal{E}(k_{\alpha}) > \mathcal{E}(\overline{k_{\alpha}}) \forall \alpha \in [\alpha_1, 1)$ .*

*If  $1 < q < 2$  and  $E'(\alpha) < 0 \forall \alpha \in (\alpha_1, 1)$  then  $\mathcal{E}(k_{\alpha}) < \mathcal{E}(\overline{k_{\alpha}}) \forall \alpha \in [\alpha_1, 1)$ .*

Note that if  $q \geq 2$  or  $q < 1$  then  $E'(\alpha) > 0$  for all  $\alpha$  by [18, Theorem 11].

*Proof of Proposition 12.* The proof depends on a number of elementary differential equations and inequalities that we derived in §§5.1–5.2 of [18], and the reader may wish to skim those sections before proceeding.

If  $q = 1$  then  $\mathcal{E}(k_{\alpha}) = \mathcal{E}(\overline{k_{\alpha}})$ , as one sees directly from the formula in (23), using that  $k_{\alpha}(x) = 1 + (\alpha - 1) \cos x$ ,  $P(\alpha) = 2\pi$  and  $\overline{k_{\alpha}} = 1$ . So we assume  $q \neq 1$  from now on, and  $\alpha \in (0, 1)$ .

By definition,

$$(29) \quad \begin{aligned} \frac{\mathcal{E}(k_{\alpha}) - \mathcal{E}(\overline{k_{\alpha}})}{P} &= \frac{1}{P} \int_0^P \left[ \frac{1}{2} k_{\alpha}'^2 - H(k_{\alpha}) \right] dx + H(A/P) \\ &= \frac{1}{P} \int_0^P k_{\alpha}'^2 dx - H(\alpha) + H(A/P) \quad \text{by [18, eq. (21)]}. \end{aligned}$$

First assume  $q \neq -1$ ; then

$$(30) \quad \frac{d}{d\alpha} \frac{\mathcal{E}(k_{\alpha}) - \mathcal{E}(\overline{k_{\alpha}})}{P} = -\frac{P'(\alpha)}{P^2} \int_0^P k_{\alpha}'^2 dx + \left( 1 + (q+1)H(A/P) \left(\frac{A}{P}\right)^{-1} \right) \left(\frac{A}{P}\right)'(\alpha),$$

where we have used [18, eq. (35)] and the identity  $H'(y) = 1 + (q+1)H(y)y^{-1}$  (valid for  $q \neq -1$ ). Differentiating the function

$$(31) \quad \mathcal{F}(\alpha) := \left(\frac{A}{P}\right)^{-(q+1)} \frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P}$$

(which is inspired by (28)) with respect to  $\alpha$ , we find from (29) and (30) that

$$\mathcal{F}'(\alpha) = \left(\frac{A}{P}\right)^{-(q+2)} \left\{ \frac{1}{P} \int_0^P k_\alpha'^2 dx \cdot \left[ -(q+1) \left(\frac{A}{P}\right)' - \frac{AP'}{P^2} \right] + \left[ \frac{A}{P} + (q+1)H(\alpha) \right] \left(\frac{A}{P}\right)' \right\}.$$

Substituting

$$\frac{A}{P} + (q+1)H(\alpha) = \frac{q+3}{2} \frac{1}{P} \int_0^P k_\alpha'^2 dx$$

from [18, eqs. (31–32)] yields

$$(32) \quad \mathcal{F}'(\alpha) = -\frac{1}{2} \left(\frac{A}{P}\right)^{-(q+2)} \frac{1}{P} \int_0^P k_\alpha'^2 dx \cdot \left[ (q-1) \left(\frac{A}{P}\right)' + 2AP^{-2}P' \right] \quad \text{for } q \neq -1.$$

For  $q = -1$  we obtain exactly the same formula (32) for  $\mathcal{F}'(\alpha)$ , as follows:

$$\begin{aligned} \mathcal{F}'(\alpha) &= \frac{d}{d\alpha} \frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P} \\ &= -\frac{P'}{P^2} \int_0^P k_\alpha'^2 dx + (1 - P/A) \left(\frac{A}{P}\right)' \quad \text{from (29) and [18, eq. (35)]} \\ &= -\frac{1}{2} \left(\frac{A}{P}\right)^{-1} \frac{1}{P} \int_0^P k_\alpha'^2 dx \cdot \left[ -2 \left(\frac{A}{P}\right)' + 2AP^{-2}P' \right] \end{aligned}$$

using that  $(1 - P/A) = \int_0^P (k_\alpha')^2 dx / A$  when  $q = -1$ , by [18, eq. (28)]. The last equation is (32) for  $q = -1$ .

Equation (32) simplifies to

$$(33) \quad \mathcal{F}'(\alpha) = - \left[ \frac{1}{2} P^{-3} \left(\frac{A}{P}\right)^{-2q} \int_0^P k_\alpha'^2 dx \right] E'(\alpha),$$

using that  $E = P^2(A/P)^{q-1}$  by (26). Hence

$$(34) \quad E' > 0 \iff \mathcal{F}' < 0 \quad \text{and} \quad E' < 0 \iff \mathcal{F}' > 0.$$

Also

$$(35) \quad \mathcal{F}(\alpha) \rightarrow 0 \quad \text{as} \quad \alpha \rightarrow 1,$$

by the formula (29) together with the facts that  $P, A \rightarrow 2\pi$  and  $P/A \rightarrow 1$  as  $\alpha \rightarrow 1$  (see [18, Lemma 6]) and that  $k_\alpha' \rightarrow 0$  uniformly as  $\alpha \rightarrow 1$ , by [18, eq. (21)]. Proposition 12 now follows from (34), (35) and [18, Theorem 11] (which shows  $E' > 0$  when  $q \geq 2$  or  $q < 1$ ). For example, if  $E' > 0$  on  $(\alpha_1, 1)$  then  $\mathcal{F}' < 0$  on  $(\alpha_1, 1)$ ; since  $\mathcal{F}(1) = 0$  we deduce  $\mathcal{F} > 0$  on  $[\alpha_1, 1)$ , and so  $\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha}) > 0$  for  $\alpha \in [\alpha_1, 1)$ .  $\square$



**5.2. Proof of Theorem 7.** The proof relies on formulas for  $A'(\alpha)$  and  $E'(\alpha)$  that were derived in Lemmas 16 and 18 of [18]: for all  $q \neq -1$ ,

$$(36) \quad A' = -(q+1)H(\alpha)P' - \frac{q-1}{2}H'(\alpha)P,$$

$$(37) \quad E' = -\left(\frac{A}{P}\right)^{q-2} \left\{ P' [(q-3)A + (q-1)(q+1)H(\alpha)P] + \frac{1}{2}(q-1)^2 H'(\alpha)P^2 \right\}.$$

Note that  $E' > 0$  if  $q < 1$  or  $q \geq 2$ , by [18, Theorem 11]. Our assumptions therefore imply  $-1 < q < 3$  and  $E'(\alpha) > 0$  for all  $\alpha \in (0, 1)$ . Also  $E(0) > 0$  because  $q > -1$  [17, §3.1.2]. If we define

$$\hat{X} = [E(0)/(\mathcal{B}A_{\text{ss}}^{q-1})]^{1/(3-q)}$$

then

$$\mathcal{B}\hat{X}^{3-q}A_{\text{ss}}^{q-1} = E(0) < E(\alpha) = \mathcal{B}X^{3-q}A_{\text{ss}}^{q-1}$$

by (26); here the value  $\alpha \in (0, 1)$  is determined by translating and rescaling  $h_{\text{ss}}$  to a particular  $k_\alpha$ , as in §2.3. Hence  $0 < \hat{X} < X$  (using that  $q < 3$ ).

By rescaling the zero contact angle function  $k_0$  on the interval  $[0, P(0)]$  (as in §2.3; see [17, Claim 5.1.2] for details) we obtain a zero contact angle droplet steady state  $\hat{h}_{\text{ss}}$  of  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ , with length  $\hat{X}$  and area  $A_{\text{ss}}$  as desired.

It remains to prove that the energy of this droplet steady state is lower than the energy of the positive periodic steady state  $h_{\text{ss}}$ . That is, we want to prove

$$\int_0^X \left[ \frac{1}{2}(\hat{h}'_{\text{ss}})^2 - \mathcal{B}G(\hat{h}_{\text{ss}}) \right] dx < \int_0^X \left[ \frac{1}{2}(h'_{\text{ss}})^2 - \mathcal{B}G(h_{\text{ss}}) \right] dx$$

where

$$G(y) = \begin{cases} \frac{y^{q+1}}{q(q+1)}, & q \neq 0, -1, \\ y \log y - y, & q = 0, \\ -\log y, & q = -1; \end{cases}$$

note that we can use  $G$  instead of  $H$  in the energy because they differ only by a linear function (cf. the definition (22) of  $H$ ) and  $\hat{h}_{\text{ss}}$  and  $h_{\text{ss}}$  have the same area,  $A_{\text{ss}}$ .

Since  $\hat{h}_{\text{ss}}$  is supported on  $(0, \hat{X})$  and because  $G(0) = 0$ , the desired inequality is

$$(38) \quad \int_0^{\hat{X}} \left[ \frac{1}{2}(\hat{h}'_{\text{ss}})^2 - \mathcal{B}G(\hat{h}_{\text{ss}}) \right] dx < \int_0^X \left[ \frac{1}{2}(h'_{\text{ss}})^2 - \mathcal{B}G(h_{\text{ss}}) \right] dx.$$

Next rescale  $\hat{h}_{\text{ss}}$  to  $k_0$  and  $h_{\text{ss}}$  to  $k_\alpha$ : from (25) and (27) (with  $P_{\text{ss}}$  replaced by  $\hat{X}$  or  $X$  as appropriate) we deduce (38) is equivalent to

$$\left[ \frac{A_{\text{ss}}}{A(0)} \right]^2 \left[ \frac{P(0)}{\hat{X}} \right]^3 \int_0^{P(0)} \left[ \frac{1}{2}(k'_0)^2 - G(k_0) \right] dx < \left[ \frac{A_{\text{ss}}}{A(\alpha)} \right]^2 \left[ \frac{P(\alpha)}{X} \right]^3 \int_0^{P(\alpha)} \left[ \frac{1}{2}(k'_\alpha)^2 - G(k_\alpha) \right] dx,$$

except that when  $q = 0$  we have to subtract

$$\left[ \log \frac{A_{\text{ss}}}{A(0)} \frac{P(0)}{\hat{X}} \right] k_0 \quad \text{and} \quad \left[ \log \frac{A_{\text{ss}}}{A(\alpha)} \frac{P(\alpha)}{X} \right] k_\alpha$$

from the integrands on the left and right sides, respectively. By substituting the relations

$$\hat{X} = [E(0)/(\mathcal{B}A_{\text{ss}}^{q-1})]^{1/(3-q)} \quad \text{and} \quad X = [E(\alpha)/(\mathcal{B}A_{\text{ss}}^{q-1})]^{1/(3-q)}$$

into the last inequality and using the definition  $E(\alpha) = P(\alpha)^{3-q}A(\alpha)^{q-1}$ , we see the desired inequality reduces to  $\mathcal{G}(0) < \mathcal{G}(\alpha)$  where (for  $q \neq 3$ )

$$(39) \quad \mathcal{G}(\alpha) = A(\alpha)^{(q+3)/(q-3)} \int_0^{P(\alpha)} \left[ \frac{1}{2}(k'_\alpha)^2 - G(k_\alpha) \right] dx \quad \left( +\frac{2}{3} \log A(\alpha), \text{ when } q = 0 \right).$$

Thus to show the energy of  $\hat{h}_{\text{ss}}$  is lower than that of  $h_{\text{ss}}$ , it suffices to show  $\mathcal{G}'(\alpha) > 0$  for all  $\alpha \in (0, 1)$ , assuming  $-1 < q < 3$  and  $E'(\alpha) > 0$  for all  $\alpha \in (0, 1)$ .

To show  $\mathcal{G}' > 0$ , we substitute [18, eq. (32)] and [18, eq. (30)] into the definition (39) of  $\mathcal{G}$ , obtaining that

$$\mathcal{G} = \frac{q-3}{q(q+3)} A^{2q/(q-3)} + \frac{q-1}{q+3} H(\alpha) P A^{(q+3)/(q-3)} \quad \text{when } q \neq 3, 0, -1, -3.$$

After differentiating this formula with respect to  $\alpha$  and then substituting for  $A'(\alpha)$  from (36), we simplify with the help of (37) to obtain

$$(40) \quad \mathcal{G}' = \frac{1}{q-3} H(\alpha) A^{6/(q-3)} \left( \frac{A}{P} \right)^{-q+2} E' \quad \text{when } q \neq 3, 0, -1, -3.$$

When  $q = 0$  we obtain  $\mathcal{G} = \frac{2}{3} - \frac{1}{3} H P A^{-1} + \frac{2}{3} \log A$ , by putting the  $q = 0$  versions of [18, eq. (32)] and [18, eq. (30)] into (39). Hence  $\mathcal{G}' = -\frac{1}{3} H P^{-2} E'$ , by differentiating and using (36) and (37). That is, (40) holds when  $q = 0$ , also.

We conclude from (40) that  $\mathcal{G}' > 0$  as desired (provided  $-1 < q < 3$  and  $E'(\alpha) > 0$  for all  $\alpha \in (0, 1)$ ), since for  $q > -1$  we have  $H(\alpha) < 0$  by the definition (22).  $\square$

**5.3. Proof of Theorem 8.** The proof involves rescaling arguments rather than the energy. Write  $P_{\text{ss}} = X$ . Assume  $\hat{h}_{\text{ss}}$  is non-constant.

Suppose that in fact  $\hat{h}_{\text{ss}}$  is positive (and so smooth by hypothesis). Then the least period of  $\hat{h}_{\text{ss}}$  equals  $P_{\text{ss}}/j$  for some positive integer  $j$ , with the area per period equaling  $A_{\text{ss}}/j$ . If  $j = 1$  then  $\hat{h}_{\text{ss}}$  must be a translate of  $h_{\text{ss}}$ , by modifying slightly the uniqueness remarks in [18, §6.2] (using the assumption that  $E' < 0$  to get strict monotonicity of  $E$ ).

Thus we can assume  $j \geq 2$ . By rescaling  $h_{\text{ss}}$  and  $\hat{h}_{\text{ss}}$  to  $k_\alpha$  and  $k_{\hat{\alpha}}$  for some  $\alpha, \hat{\alpha} \in (0, 1)$ , as in (7), we get from (26) that

$$E(\alpha) = \mathcal{B} P_{\text{ss}}^{3-q} A_{\text{ss}}^{q-1} \quad \text{and} \quad E(\hat{\alpha}) = \mathcal{B} (P_{\text{ss}}/j)^{3-q} (A_{\text{ss}}/j)^{q-1}.$$

Hence

$$\begin{aligned} 4 \leq j^2 = \frac{E(\alpha)}{E(\hat{\alpha})} &< \frac{E(0)}{E(1)} \quad \text{since } E' < 0 \\ &= \frac{1}{4\pi^2} \frac{2}{q} (1+q) B\left(\frac{1}{2q}, \frac{1}{2}\right)^{3-q} B\left(\frac{3}{2q}, \frac{1}{2}\right)^{q-1} =: J(q) \text{ say,} \end{aligned}$$

by the formula for  $E(0)$  in [17, eq. (3.13)] and since  $E(1) = P(1)^{3-q} A(1)^{q-1} = 4\pi^2$  by [18, Lemma 6]. We will obtain a contradiction by showing  $J(q) < 4$ , when  $1 < q \leq 1.75$ ; this will show  $\hat{h}_{ss}$  is not positive.

For  $1 < q \leq 1.5$  we have

$$J(q) \leq \frac{1}{4\pi^2} \frac{2}{1} (1+1.5) B\left(\frac{1}{2 \cdot 1.5}, \frac{1}{2}\right)^{3-1} B\left(\frac{3}{2 \cdot 1.5}, \frac{1}{2}\right)^{1.5-1} \approx 3.17 < 4,$$

where we have used that the Beta function  $B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$  is decreasing in its arguments, and is bigger than 1 when those arguments are less than 1. For  $1.5 < q \leq 1.75$  we similarly have

$$J(q) \leq \frac{1}{4\pi^2} \frac{2}{1.5} (1+1.75) B\left(\frac{1}{2 \cdot 1.75}, \frac{1}{2}\right)^{3-1.5} B\left(\frac{3}{2 \cdot 1.75}, \frac{1}{2}\right)^{1.75-1} \approx 1.73 < 4,$$

completing the contradiction. (Actually the best upper bound for  $J(q)$  seems numerically to be about 1.04, when  $1 < q \leq 1.75$ ; see [18, Figure 6].)

The above contradiction implies  $\hat{h}_{ss}$  is not positive everywhere. Consider therefore one component of the set  $\{x : \hat{h}_{ss}(x) > 0\}$ , say an interval with length  $\hat{P} \leq P_{ss}$ . Write  $\hat{A} \leq A_{ss}$  for the area under  $\hat{h}_{ss}$  on this interval. Note the contact angles of  $\hat{h}_{ss}$  must be the same at the two endpoints of the interval, as a consequence of the nonlinear oscillator equation (6) (see for example [17, §2.2]). Suppose these contact angles are zero, so that  $\hat{h}_{ss}$  rescales to  $k_0$  on the interval, using (7). Then

$$E(0) = \mathcal{B} \hat{P}^{3-q} \hat{A}^{q-1} \leq \mathcal{B} P_{ss}^{3-q} A_{ss}^{q-1} = E(\alpha)$$

by applying (26) twice, but this contradicts our assumption that  $E' < 0$ . Thus the contact angles of  $\hat{h}_{ss}$  must all be nonzero, as desired.  $\square$

**5.4. Proof of Theorem 9.** Translate  $h_{ss1}$  and  $h_{ss2}$  so that they attain their minimum values at  $x = 0$ , and then rescale as in §2.3 to obtain  $k_{\alpha_1}$  and  $k_{\alpha_2}$  respectively. Since  $h_{ss1}$  and  $h_{ss2}$  have the same period and area, for which we write  $P_{ss} = X$  and  $A_{ss}$  respectively, it follows from (26) that  $E(\alpha_1) = E(\alpha_2)$ . Notice  $h_{ss1} \neq h_{ss2} \Rightarrow \alpha_1 \neq \alpha_2$ , in view of the expression (25) for  $h_{ss}$  in terms of  $k_\alpha$  and  $P_{ss}, A_{ss}, P(\alpha), A(\alpha)$ . Since  $E(\alpha_1) = E(\alpha_2)$  while  $E$  is strictly decreasing on  $(0, \alpha_{crit})$  and strictly increasing on  $(\alpha_{crit}, 1)$ , we conclude  $\alpha_{crit}$  must lie between  $\alpha_1$  and  $\alpha_2$ .

We show  $\alpha_1 < \alpha_2$ . The hypothesis  $\min h_{ss1} < \min h_{ss2}$  gives

$$\alpha_1 \left( \frac{D_1}{\mathcal{B}} \right)^{1/q} < \alpha_2 \left( \frac{D_2}{\mathcal{B}} \right)^{1/q}$$

by the rescaling (7). Next apply the first equation in (24) to solve for  $D_1$  in terms of  $P(\alpha_1)$ ,  $P_{ss}$  and  $\mathcal{B}$ , and similarly solve for  $D_2$  in terms of  $P(\alpha_2)$ ,  $P_{ss}$  and  $\mathcal{B}$ . Substituting into the above inequality gives  $\alpha_1 P(\alpha_1)^{2/(q-1)} < \alpha_2 P(\alpha_2)^{2/(q-1)}$ . The strict increase of  $\alpha \mapsto \alpha P(\alpha)^{2/(q-1)}$  implies  $\alpha_1 < \alpha_2$ .

Since  $\alpha_1 < \alpha_{crit} < \alpha_2$ , our hypothesis on  $E'$  implies  $E'(\alpha_1) < 0$  and  $E'(\alpha_2) > 0$ . Theorem 3 then implies that  $h_{ss1}$  is energy stable, and Theorem 2 implies  $h_{ss2}$  is energy unstable.

Next we show  $\mathcal{E}(h_{ss1}) < \mathcal{E}(h_{ss2})$ , or  $\mathcal{E}(h_{ss1}) - \mathcal{E}(\overline{h_{ss1}}) < \mathcal{E}(h_{ss2}) - \mathcal{E}(\overline{h_{ss2}})$ . In view of the rescaling relation (28) for the energy, it suffices to prove  $\mathcal{F}(\alpha_1) < \mathcal{F}(\alpha_2)$ , where  $\mathcal{F}$  was defined in (31).

To prove  $\mathcal{F}(\alpha_1) < \mathcal{F}(\alpha_2)$ , we write (33) as  $\mathcal{F}'(\alpha) = \delta(\alpha)(1/E)'(\alpha)$ , where

$$\delta(\alpha) = \frac{1}{2} P^3 A^{-2} \int_0^P k'_\alpha{}^2 dx.$$

The point of this transformation is that  $\delta(\alpha)$  is strictly decreasing:  $P' < 0$  and  $A' > 0$ , by [17, Props. 7.3 and 7.4], while  $\alpha \mapsto \int_0^P k'_\alpha{}^2 dx$  is decreasing by [18, eq. (35)]. Also  $(1/E)' > 0$  on  $(0, \alpha_{crit})$  and  $(1/E)' < 0$  on  $(\alpha_{crit}, 1)$ , by assumption. Hence

$$\mathcal{F}'(\alpha) > \delta(\alpha_{crit})(1/E)'(\alpha) \quad \text{for } \alpha \in (0, 1), \alpha \neq \alpha_{crit}.$$

Integrating this inequality from  $\alpha_1$  to  $\alpha_2$  and using that  $E(\alpha_1) = E(\alpha_2)$  gives  $\mathcal{F}(\alpha_2) > \mathcal{F}(\alpha_1)$ , as desired.

Finally, (33) shows  $\mathcal{F}'(\alpha) < 0$  on  $(\alpha_{crit}, 1)$ , and so  $\mathcal{F}(\alpha_2) > \mathcal{F}(1) = 0$  by (35). Thus (28) yields  $\mathcal{E}(h_{ss2}) > \mathcal{E}(\overline{h_{ss2}})$ .  $\square$

**5.5. Proof of Theorem 10.** From the definition (11) we see

$$\tau_1(\overline{h}) = \min_u \frac{\int_0^X (u')^2 dx}{\int_0^X u^2 dx} - r(\overline{h}) = \left( \frac{2\pi}{X} \right)^2 - r(\overline{h}),$$

with the minimum being attained precisely for linear combinations of  $\sin(2\pi x/X)$  and  $\cos(2\pi x/X)$ . The first two paragraphs of the theorem follow directly.

(a) Now suppose  $r(\overline{h})X^2 > 4\pi^2$ . The variational formulas (15) and (16) in the proof of Theorem 1 show the constant steady state  $\overline{h}$  is energy unstable in the sine and cosine directions, since these are  $\tau_1(\overline{h})$ -eigenfunctions and  $\tau_1(\overline{h}) < 0$ . Suppose next  $r(\overline{h})X^2 = 4\pi^2$  and  $r''(\overline{h}) > 0$ . Then the first two variations of the energy in the  $\pm \sin$  directions are zero, by (15) and (16). The third variation equals  $r'(\overline{h})$  times the integral of  $\mp \sin^3$ , by (17); thus the third variation is also zero. The fourth variation is  $-r''(\overline{h}) \int_0^X \sin^4(2\pi x/X) dx$ , which is negative because we assumed  $r''(\overline{h}) > 0$ . Thus the constant steady state is energy unstable in the  $\pm \sin$  directions. Argue similarly for the  $\pm \cos$  directions.

(b) If  $r(\bar{h})X^2 < 4\pi^2$ , or if  $r(\bar{h})X^2 = 4\pi^2$  and  $r''(\bar{h}) < 0$ , then  $\tau_1(\bar{h}) \geq 0$  and so we get energy stability by modifying the argument of part (a) as follows. The first variation of the energy in a direction  $u$  is always zero. If the second variation is positive then we are done. Otherwise the second variation must be zero, so that  $\tau_1(\bar{h}) = 0$  and  $r''(\bar{h}) < 0$ . Then  $u$  must be a linear combination of sines and cosines, and so the third variation is also zero. Then the fourth variation is positive.

Furthermore, for  $r(\bar{h})X^2 < 4\pi^2$  we will prove  $\bar{h}$  is a strict local minimum of the energy, and is nonlinearly stable in  $H^1$ . In doing this, we will use below a certain sufficiently small number  $\delta \in (0, 1)$ . Then considering  $u \in H^1(\mathbb{T}_X)$  with mean value zero and  $\|u\|_{H^1(\mathbb{T}_X)} = 1$ , we find for all  $\varepsilon \in [0, \delta]$  that

$$\begin{aligned} \frac{d^2}{d\varepsilon^2} \mathcal{E}(\bar{h} + \varepsilon u) &= \int_0^X [(u')^2 - r(\bar{h} + \varepsilon u)u^2] dx \\ &= \delta \int_0^X [(u')^2 + u^2] dx + (1 - \delta) \int_0^X \left[ (u')^2 - \frac{r(\bar{h} + \varepsilon u) + \delta}{1 - \delta} u^2 \right] dx \\ &> \delta \|u\|_{H^1(\mathbb{T}_X)}^2 + (1 - \delta) \int_0^X \left[ (u')^2 - \frac{4\pi^2}{X^2} u^2 \right] dx \geq \delta \|u\|_{H^1(\mathbb{T}_X)}^2. \end{aligned}$$

In the second-to-last step above, we used  $r(\bar{h})X^2 < 4\pi^2$  and  $\|u\|_\infty \leq C\|u\|_{H^1}$  and we chose  $\delta$  sufficiently small (independent of  $u$  and  $\varepsilon$ ).

On the other hand,  $|\varepsilon u| \leq \delta \|u\|_{L^\infty} \leq \bar{h}/2$  provided  $\delta$  is chosen small enough (independent of  $u$ ), and hence

$$\begin{aligned} \frac{d^2}{d\varepsilon^2} \mathcal{E}(\bar{h} + \varepsilon u) &= \int_0^X [(u')^2 - r(\bar{h} + \varepsilon u)u^2] dx \leq C(\bar{h}) \int_0^X [(u')^2 + u^2] dx \\ &\quad \text{for some constant } C(\bar{h}) \geq 1 \\ &= C(\bar{h}) \|u\|_{H^1(\mathbb{T}_X)}^2. \end{aligned}$$

We deduce from the preceding estimates and the vanishing of the first variation in (15) that if  $u \in H^1(\mathbb{T}_X)$  has mean value zero and  $\|u\|_{H^1(\mathbb{T}_X)} \leq \delta$ , then  $\bar{h} + u > 0$  and the energy varies quadratically away from  $\bar{h}$ :

$$(41) \quad \frac{1}{2} \delta \|u\|_{H^1(\mathbb{T}_X)}^2 \leq \mathcal{E}(\bar{h} + u) - \mathcal{E}(\bar{h}) \leq \frac{1}{2} C(\bar{h}) \|u\|_{H^1(\mathbb{T}_X)}^2.$$

The lefthand estimate implies  $\bar{h}$  is a strict local minimum of the energy, with respect to  $X$ -periodic zero-mean perturbations.

We now prove  $\bar{h}$  is nonlinearly stable, in the sense that if  $h(x, t)$  is a smooth positive solution of (1) for  $x \in \mathbb{T}_X$  and  $t \in [0, T]$ , for some  $T > 0$ , and if  $h(\cdot, 0)$  has mean value  $\bar{h}$  and  $\|h(\cdot, 0) - \bar{h}\|_{H^1(\mathbb{T}_X)} < \sqrt{\delta^3/4C(\bar{h})}$ , then  $\|h(\cdot, t) - \bar{h}\|_{H^1(\mathbb{T}_X)} < \delta/2$  for all  $t \in [0, T]$ . Indeed,

the quadratic bounds (41) and the dissipation of the energy together imply

$$\begin{aligned} \frac{1}{2}\delta\|h(\cdot, t) - \bar{h}\|_{H^1(\mathbb{T}_X)}^2 &\leq \mathcal{E}(h(\cdot, t)) - \mathcal{E}(\bar{h}) \\ &\leq \mathcal{E}(h(\cdot, 0)) - \mathcal{E}(\bar{h}) \leq \frac{1}{2}C(\bar{h})\|h(\cdot, 0) - \bar{h}\|_{H^1(\mathbb{T}_X)}^2 < \frac{1}{2}\frac{\delta^3}{4}, \end{aligned}$$

so that  $\|h(\cdot, t) - \bar{h}\|_{H^1(\mathbb{T}_X)} < \delta/2$  for all  $t \in [0, T]$ . This stability result holds for all sufficiently small  $\delta$ .  $\square$

This proof of nonlinear stability for a linearly stable constant steady state does not carry over to linearly stable positive periodic steady states  $h_{ss}$ , because there  $\tau_1(h_{ss}) = 0$  (due to translational null directions). This zero eigenvalue is absent for the constant steady state, since translation of  $\bar{h}$  gives  $\bar{h}$  again — a trivial perturbation. Imposing Neumann boundary conditions eliminates the translational perturbations and their associated zero eigenvalue. Hence, the nonlinear stability proof *would* hold for a positive steady state under the Neumann boundary conditions  $h_x = h_{xxx} = 0$  at  $x = 0, X/2$  (see §2.6), provided the steady state is strictly linearly stable, *i.e.* the first eigenvalue is positive.

**5.6. Proof of Theorem 11.** Suppose first that  $-1 < q < 3$ . By (26) with  $\alpha = 0$ , a steady state  $\hat{h}_{ss}$  supported on a single interval of length  $\hat{X} \leq X$  and with area  $\hat{A}_{ss} = \bar{h}X$  and zero contact angles exists if and only if for some length  $\hat{X} \leq X$  we have

$$(42) \quad \mathcal{B}\hat{X}^{3-q}\hat{A}_{ss}^{q-1} = E(0) = P(0)^{3-q}A(0)^{q-1}.$$

That is, if and only if  $\mathcal{B}X^{3-q}(\bar{h}X)^{q-1} \geq E(0)$ , which is (12).

Suppose such a steady state  $\hat{h}_{ss}$  exists, supported say on the interval  $(0, \hat{X})$ . By above,

$$\hat{X} = \mathcal{B}^{1/(q-3)}P(0)[A(0)/\bar{h}X]^{(q-1)/(3-q)}.$$

We want to show that  $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(\bar{h})$  if and only if (13) holds. For this, compute using  $G$  like in §5.2 to find

$$\begin{aligned} \mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\bar{h}) &= \int_0^X \left[ \frac{1}{2} \left( \hat{h}'_{ss} \right)^2 - \mathcal{B}G(\hat{h}_{ss}) + \mathcal{B}G(\bar{h}) \right] dx \\ &= \int_0^{\hat{X}} \left[ \frac{1}{2} \left( \hat{h}'_{ss} \right)^2 - \mathcal{B}G(\hat{h}_{ss}) \right] dx + \mathcal{B}G(\bar{h})X \\ (43) \quad &= \frac{\hat{A}_{ss}^2 P(0)^3}{A(0)^2 \hat{X}^3} \int_0^{P(0)} \left[ \frac{1}{2} (k'_0)^2 - G(k_0) \right] dx + \mathcal{B}G(\bar{h})X \end{aligned}$$

when  $q > -1, q \neq 0$ , by rescaling from  $\hat{h}_{ss}$  to  $k_0$  and using (25) and (27) (with  $P_{ss}$  replaced by  $\hat{X}$ , and  $A_{ss}$  replaced by  $\hat{A}_{ss}$ ). Putting  $\alpha = 0$  into [18, eq. (32)] and [18, eq. (30)] and using  $H(0) = 0$  enables us to evaluate  $\int (k'_0)^2 dx$  and  $\int G(k_0) dx$ , and hence we deduce

$$(44) \quad \mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\bar{h}) = \frac{\hat{A}_{ss}^2 P(0)^3}{A(0)^2 \hat{X}^3} \frac{q-3}{q(q+3)} A(0) + \mathcal{B}G(\bar{h})X \quad \text{when } q > -1, q \neq 0.$$

After substituting the definitions of  $\hat{X}$  and  $\hat{A}_{ss} = \bar{h}X$  from above, we find

$$\mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\bar{h}) = \frac{1}{q} \mathcal{B}^{3/(3-q)} (\bar{h}X)^{(3+q)/(3-q)} \left[ \frac{q-3}{q+3} A(0)^{2q/(q-3)} + \frac{1}{q+1} \left( \mathcal{B} \bar{h}^{q-1} X^2 \right)^{q/(q-3)} \right]$$

when  $-1 < q < 3, q \neq 0$ . Plainly now  $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(\bar{h})$  if and only if (13) holds.

When  $q = 0$  we find (43) has an extra term  $-\mathcal{B} \hat{A}_{ss} \log [\hat{A}_{ss} P(0)/A(0) \hat{X}]$ , so that after again using [18, eq. (32)] and [18, eq. (30)] and substituting for  $\hat{X}$  and  $\hat{A}_{ss}$ , we obtain

$$\mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\bar{h}) = \frac{1}{3} \mathcal{B} \bar{h} X \log \frac{A(0)^2/e}{\mathcal{B} \bar{h}^{-1} X^2} \quad \text{when } q = 0.$$

Remembering that  $A(0) = 2e^{3/2} \sqrt{\pi/3}$  when  $q = 0$ , from [17, §3.1.2], we conclude  $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(\bar{h})$  if and only if  $\mathcal{B} \bar{h}^{-1} X^2 > 4e^2 \pi/3$ . Incidentally, one can check  $L(q)$  is continuous at  $q = 0$ .

For  $q \geq 3$ , simply modify the above proof from the case  $-1 < q < 3$ . Notice when  $q = 3$  that (42) becomes  $\mathcal{B}(\bar{h}X)^2 = E(0)$ , which yields no formula for  $\hat{X}$ . And when  $q \geq 3$  we get  $\mathcal{E}(\hat{h}_{ss}) > \mathcal{E}(\bar{h})$  because the first term in (44) is nonnegative and the second is positive.  $\square$

## 6. CONCLUSIONS AND FUTURE DIRECTIONS

If you perturb a positive periodic (or constant) steady state  $h_{ss}$  of the evolution equation (1), without changing its area, then towards which steady states might the solution subsequently evolve?

That is the broad question addressed by this paper. To answer it, we focused on three specific questions:

- **EXISTENCE:** Do there exist *other* steady states having the same area and same period as  $h_{ss}$ , or having period a *fraction* of the period of  $h_{ss}$ ? If so, then these other steady states are plausible contenders for the long time limit. The constant steady state  $\bar{h}_{ss}$  obviously satisfies these requirements, but there might be another positive periodic steady state (different from  $h_{ss}$ ) that does also, or perhaps a array of droplet steady states having zero or nonzero contact angles.
- **STABILITY:** Are  $h_{ss}$  or these other steady states linearly stable? energy stable? If a steady state is the long time limit of some generic solution, then surely it must be stable under perturbations.
- **RELATIVE ENERGY LEVELS:** Do any of these other steady states have lower energy than  $h_{ss}$ ? Only steady states with lower energy are accessible, when starting from a small perturbation of  $h_{ss}$ .

The EXISTENCE question was substantially answered for power law coefficients by [18, Theorem 12], Theorems 7 and 8 and [18, Figures 3–5], also using Theorem 11 when  $h_{ss}$  is the constant steady state. But the existence question remains open for droplet steady states

with *nonzero* contact angles, if we wish to specify the area and the length of the support. Some information on nonzero angle droplet steady states is in our earlier paper [17, §5.2].

The STABILITY question was resolved for positive periodic steady states in the power law case by [18, Theorems 1,3,7,9] and Theorems 1–5 here. In particular, Theorem 1 related linear instability to energy instability. Theorem 10 handled the case of constant steady states. Our numerical simulations in the companion article [19] suggest that linearly unstable steady states are indeed nonlinearly unstable, with the linear behavior dominating near the steady state, but this observation is so far unsupported by a ‘linearization theorem’ for the power law evolution  $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ . (A linearization theorem *is* known in the Cahn–Hilliard case  $f \equiv 1$ , by using semilinear operator theory; see for example [23, §6].)

The ENERGY LEVEL question has been largely settled in the power law case by Theorems 6, 7 and 9 when  $h_{ss}$  is positive and periodic, and by Theorems 6 and 11 when  $h_{ss}$  is constant. When  $h_{ss}$  has compact support with zero contact angle, use Theorems 7 and 11. For example, Theorems 9 suggests that when  $m = n + 0.77$ , there can exist two positive periodic steady states with the same period and area, and that the unstable steady state has higher energy than the stable one. Our numerical simulations have found heteroclinic connections from the high energy steady state to the low energy one.

**Future directions.** The stability question for *droplet* steady states (with zero and nonzero contact angles) is open. So is the problem of computing relative energy levels of non-zero angle droplet steady states with respect to zero-angle droplets and constant and periodic steady states.

Another open problem is to answer the existence, stability and energy level questions for *general* coefficient functions  $f$  and  $g$ . We have treated power law coefficients, and Grinfeld and Novick–Cohen [14] cover the Cahn–Hilliard equation, with  $f \equiv 1$  and  $g(y) = 1 - 3y^2$ . But for general coefficients, about all we know is that every positive periodic steady state is linearly and energy unstable when  $g/f$  is a convex function, by Theorem 1.

Finally, many of our existence, stability and relative energy level theorems for the power law evolution would be improved if we knew  $E'(\alpha) < 0$  for all  $\alpha$  when  $1 < q \leq 1.75$ . We have not been able to prove this conjecture, though numerically it is clear from [18, Figure 3].

**Acknowledgments.** The authors are grateful to Andrew Bernoff for stimulating comments on the energy landscape of phase space.

Laugesen was partially supported by NSF grant number DMS-9970228, and a grant from the University of Illinois Research Board. He is grateful for the hospitality of the Department of Mathematics at Washington University in St. Louis.

Pugh was partially supported by NSF grant number DMS-9971392, by the MRSEC Program of the NSF under Award Number DMR-9808595, by the ASCI Flash Center at the



University of Chicago under DOE contract B341495, and by an Alfred P. Sloan fellowship. Some of the computations were done using a network of workstations paid for by an NSF SCREMS grant, DMS-9872029. Part of the research was conducted while enjoying the hospitality of the Mathematics Department and the James Franck Institute of the University of Chicago.

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